

Non-renormalization theorem in a lattice supersymmetric theory and the cyclic Leibniz rule

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ABSTRACT: $N = 4$ supersymmetric quantum mechanical model is formulated on the lattice. Two supercharges, among four, are exactly conserved with the help of the cyclic Leibniz rule without spoiling the locality. In use of the cohomological argument, any possible local terms of the effective action are classified into two categories which we call type-I and type-II, analogous to the D- and F-terms in the supersymmetric field theories. We prove non-renormalization theorem on the type-II terms which include mass and interaction terms with keeping a lattice constant finite, while type-I terms such as the kinetic terms have nontrivial quantum corrections.

KEYWORDS: lattice supersymmetry, Leibniz rule, non-renormalization.

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1. Introduction

Supersymmetry is not only a fascinating idea for solving the gauge hierarchy problem, but also has many interesting properties worth investigating in its own light, one of which is nonrenormalization of F -terms [1, 2] in the Wess-Zumino model. In order to investigate such non-trivial properties by non-perturbative means, a lattice formulation has been desired for a long time [3, 4]. It is, however, difficult due to the lack of the Leibniz rule of finite difference operators on the lattice [5, 6].

In our previous paper, we proposed a novel approach where we use a finite difference operator and field products which satisfy *cyclic Leibniz rule* (CLR) instead of ordinary *Leibniz rule* (LR) [7, 8, 9]. Both rules coincide with each other in the continuum limit,

i.e. they reduce to the Leibniz rule of differential operator. With the CLR approach we realized a lattice quantum mechanical model in which kinetic and interaction terms are supersymmetric invariant separately, so that we succeeded to apply localization technique to obtain some exact results of the model. An important point we should emphasize here is that there indeed exist concrete sets of finite difference operator and field product which satisfy locality, translational invariance and the CLR. A systematic method for getting a general solution of the CLR is recently found in [10].

In the present paper, we proceed further to realize more supersymmetries by our approach. We consider $N = 4$ supersymmetric quantum mechanical model which is obtained by dimensional reduction of $N = 2$ Wess-Zumino model in two dimensions. We will construct its lattice version which exactly preserve two supersymmetry among four thanks to the CLR. As far as our knowledge is concerned, this is the first example which exactly realize two independent supersymmetries in the lattice model. Two exactly realized supersymmetric charges satisfy maximal nilpotent subalgebra, and surprisingly are sufficient to lead us to the non-renormalization theorem in the lattice model.

Let us note that if we use Wilson term to avoid doubling problem, the term must also keep the same number of supersymmetries. In our approach Wilson term as well as ordinary mass term is constructed in use of the CLR, so that they keep all the necessary supersymmetries and are protected by the non-renormalization theorem from quantum corrections.

The paper is organized as follows. In the next section, we first construct a lattice quantum mechanical model with two exact supersymmetries with the help of the CLR. There we introduce a kind of superfields which are slightly different from conventional ones, but useful for the subsequent discussions such as non-renormalization theorem. We then discuss on the one-particle irreducible effective action including quantum correction. Since our model contains all the necessary auxiliary fields, supersymmetries are realized off-shell and the fields are transformed linearly. Thus supersymmetry transformations of the field variables in the effective action have the same form as the elementary field variables in the tree action. This fact means that the difference operator itself appeared in the transformation has no quantum correction. Since the realized symmetry is a maximal nilpotent subalgebra of the $N = 4$ supersymmetry, we utilize cohomological argument for the classification of the supersymmetric invariant terms which is discussed in section 3. There are two categories which we call type-I and type-II each of which is an analogue of D- and F-term in the Wess-Zumino model. In section 4, we will see that this model is not a trivial one by showing one-loop quantum corrections for the type-I terms such as the kinetic term. In section 5, we prove non-renormalization theorem for the type-II terms without taking continuum limit. Section 6 is devoted to the summary and discussion. Some useful definitions and formulae are given in appendices.

2. $N = 4$ supersymmetric complex quantum mechanics

2.1 Supersymmetric transformations and superfields

We consider a lattice version of complex quantum mechanical model with $N = 4$ super-

symmetry. Our model has four sets of complex fundamental lattice fields¹ (i.e. eight real degrees of freedom):

$$\chi_{\pm n}, \bar{\chi}_{\pm n}, \phi_{\pm n}, F_{\pm n}, \quad (2.1)$$

where ϕ_{\pm}, F_{\pm} ($\chi_{\pm}, \bar{\chi}_{\pm}$) are bosonic (fermionic) variables, and n stands for a lattice site. The lattice constant is set to be unity for brevity. An inner product between these fields is defined as $A \cdot B = \langle A, B \rangle \equiv \sum_n A_n B_n$.

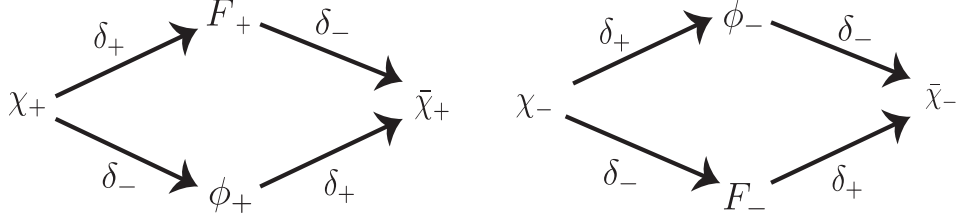


Figure 1: δ_{\pm} transformation for fields

These lattice fields are transformed under two supersymmetries δ_{\pm} as follows:

$$\left\{ \begin{array}{l} \delta_+ \phi_{+n} = \bar{\chi}_{+n}, \\ \delta_+ \bar{\chi}_{+n} = 0, \\ \delta_+ \chi_{+n} = F_{+n}, \\ \delta_+ F_{+n} = 0, \\ \delta_+ \phi_{-n} = 0, \\ \delta_+ \bar{\chi}_{-n} = 0, \\ \delta_+ \chi_{-n} = -i(\nabla \phi_-)_n, \\ \delta_+ F_{-n} = -i(\nabla \bar{\chi}_-)_n, \end{array} \right. \quad \left\{ \begin{array}{l} \delta_- \phi_{-n} = -\bar{\chi}_{-n}, \\ \delta_- \bar{\chi}_{-n} = 0, \\ \delta_- \chi_{-n} = F_{-n}, \\ \delta_- F_{-n} = 0, \\ \delta_- \phi_{+n} = 0, \\ \delta_- \bar{\chi}_{+n} = 0, \\ \delta_- \chi_{+n} = i(\nabla \phi_+)_n, \\ \delta_- F_{+n} = -i(\nabla \bar{\chi}_+)_n, \end{array} \right. \quad (2.2)$$

where ∇ is a local difference operator on a field such as $(\nabla \phi)_m \equiv \sum_n \nabla_{mn} \phi_n$. These transformations are diagrammatically depicted in figure 1. As we will see later, (2.2) is exactly realized in our lattice theory. The remaining transformations in full $N = 4$ SUSY are

$$\left\{ \begin{array}{l} \bar{\delta}_+ \phi_{+n} = 0, \\ \bar{\delta}_+ \bar{\chi}_{+n} = -i(\nabla \phi_+)_n, \\ \bar{\delta}_+ \chi_{+n} = 0, \\ \bar{\delta}_+ F_{+n} = -i(\nabla \chi_+)_n, \\ \bar{\delta}_+ \phi_{-n} = \chi_{-n}, \\ \bar{\delta}_+ \bar{\chi}_{-n} = F_{-n}, \\ \bar{\delta}_+ \chi_{-n} = 0, \\ \bar{\delta}_+ F_{-n} = 0, \end{array} \right. \quad \left\{ \begin{array}{l} \bar{\delta}_- \phi_{-n} = 0, \\ \bar{\delta}_- \bar{\chi}_{-n} = i(\nabla \phi_-)_n, \\ \bar{\delta}_- \chi_{-n} = 0, \\ \bar{\delta}_- F_{-n} = -i(\nabla \chi_-)_n, \\ \bar{\delta}_- \phi_{+n} = -\chi_{+n}, \\ \bar{\delta}_- \bar{\chi}_{+n} = F_{+n}, \\ \bar{\delta}_- \chi_{+n} = 0, \\ \bar{\delta}_- F_{+n} = 0. \end{array} \right. \quad (2.3)$$

¹we use a term “field” for a dynamical variable of the model because it can be regarded as 0+1 dimensional field theory.

The SUSY transformations $\delta_{\pm}, \bar{\delta}_{\pm}$ are realized by the following differential operators,

$$\begin{aligned}
Q_+ &\equiv \bar{\chi}_+ \cdot \frac{\partial}{\partial \phi_+} + F_+ \cdot \frac{\partial}{\partial \chi_+} - i\nabla \phi_- \cdot \frac{\partial}{\partial \chi_-} - i\nabla \bar{\chi}_- \cdot \frac{\partial}{\partial F_-}, \\
Q_- &\equiv i\nabla \phi_+ \cdot \frac{\partial}{\partial \chi_+} - i\nabla \bar{\chi}_+ \cdot \frac{\partial}{\partial F_+} - \bar{\chi}_- \cdot \frac{\partial}{\partial \phi_-} + F_- \cdot \frac{\partial}{\partial \chi_-}, \\
\bar{Q}_+ &\equiv -i\nabla \phi_+ \cdot \frac{\partial}{\partial \bar{\chi}_+} - i\nabla \chi_+ \cdot \frac{\partial}{\partial F_+} + \chi_- \cdot \frac{\partial}{\partial \phi_-} + F_- \cdot \frac{\partial}{\partial \bar{\chi}_-}, \\
\bar{Q}_- &\equiv -\chi_+ \cdot \frac{\partial}{\partial \phi_+} + F_+ \cdot \frac{\partial}{\partial \bar{\chi}_+} + i\nabla \phi_- \cdot \frac{\partial}{\partial \bar{\chi}_-} - i\nabla \chi_- \cdot \frac{\partial}{\partial F_-}.
\end{aligned} \tag{2.4}$$

From (2.4), we find the following anti-commutation relations as a part of the $N = 4$ SUSY algebra,

$$\{Q_{\pm}, \bar{Q}_{\mp}\} = \{Q_{\pm}, Q_{\pm}\} = \{Q_{\pm}, Q_{\mp}\} = \{\bar{Q}_{\pm}, \bar{Q}_{\pm}\} = \{\bar{Q}_{\pm}, \bar{Q}_{\mp}\} = 0, \tag{2.5}$$

which could be realized on lattice. The last piece of the algebra is given by

$$\{Q_{\pm}, \bar{Q}_{\pm}\} = P, \quad \text{where } iP \equiv \sum_{f=\phi_{\pm}, F_{\pm}, \chi_{\pm}, \bar{\chi}_{\pm}} (\nabla f) \cdot \frac{\partial}{\partial f}. \tag{2.6}$$

The problem is that P cannot be realized as an exact symmetry on the lattice, because it does not satisfy the relation $PX_n = -i(\nabla X)_n$ for general composite fields X_n . This comes from the fact that the Leibniz rule of a finite difference operator cannot be realized on the lattice due to the no-go theorem[5, 6]. Therefore we must try to realize only a part of nilpotent subalgebra (2.5). A possible maximal set of supercharges is either (Q_+, Q_-) or (\bar{Q}_+, \bar{Q}_-) .²

For concreteness we take

$$Q_+^2 = Q_-^2 = \{Q_+, Q_-\} = 0, \tag{2.7}$$

as a maximal nilpotent subalgebra to be realized in our model. From here on we call this nilpotent-SUSY. Our task is now to find functionals consisting of fundamental fields which satisfy

$$Q_{\pm} \mathcal{O} = 0. \tag{2.8}$$

To this end, it is helpful to examine cohomology of the nilpotent supercharges as will be seen shortly.

For later convenience, we assign a $U(1)$ charge ± 1 to the fields with \pm index respectively. In addition to this, we assign another $U(1)$ charge (we call it $U(1)_R$) which is defined by the eigenvalue of each field to the operator

$$R \equiv \phi_+ \cdot \frac{\partial}{\partial \phi_+} - F_+ \cdot \frac{\partial}{\partial F_+} - \phi_- \cdot \frac{\partial}{\partial \phi_-} + F_- \cdot \frac{\partial}{\partial F_-}. \tag{2.9}$$

²There are other choices (Q_+, \bar{Q}_-) and (\bar{Q}_+, Q_-) whose algebraic structures are the same as in the main text. Their realizations, however, are different and the following arguments do not apply straightforwardly.

In the holomorphy-like argument in later section, we will assign these $U(1)$ and $U(1)_R$ charges also to the parameters in the action such as coupling constants as well as mass parameter and the Wilson coefficient (a coefficient of the Wilson term). The quantum numbers (fermion number N_F , $U(1)$ and $U(1)_R$ charges) for fields are summarized in Table 1.

Table 1: Quantum numbers for lattice fields

	ϕ_{\pm}	$\bar{\chi}_{\pm}$	χ_{\pm}	F_{\pm}
N_F	0	-1	1	0
$U(1)$	± 1	± 1	± 1	± 1
$U(1)_R$	± 1	0	0	∓ 1

We define following additional operators

$$\begin{aligned} K_+ &= \phi_+ \cdot \frac{\partial}{\partial \bar{\chi}_+} + \chi_+ \cdot \frac{\partial}{\partial F_+}, \\ K_- &= -\phi_- \cdot \frac{\partial}{\partial \bar{\chi}_-} + \chi_- \cdot \frac{\partial}{\partial F_-}, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} N_+ &\equiv \phi_+ \cdot \frac{\partial}{\partial \phi_+} + \bar{\chi}_+ \cdot \frac{\partial}{\partial \bar{\chi}_+} + \chi_+ \cdot \frac{\partial}{\partial \chi_+} + F_+ \cdot \frac{\partial}{\partial F_+}, \\ N_- &\equiv \phi_- \cdot \frac{\partial}{\partial \phi_-} + \bar{\chi}_- \cdot \frac{\partial}{\partial \bar{\chi}_-} + \chi_- \cdot \frac{\partial}{\partial \chi_-} + F_- \cdot \frac{\partial}{\partial F_-}. \end{aligned} \quad (2.11)$$

The $U(1)$ charge defined before is equivalent to the eigenvalue of the operator $N_+ - N_-$. These operators and supercharges Q_{\pm} satisfy the following algebra

$$\begin{aligned} \{Q_{\pm}, K_{\pm}\} &= N_{\pm}, \quad [Q_{\pm}, R] = \pm Q_{\pm}, \quad [K_{\pm}, R] = \mp K_{\pm}, \\ [Q_{\pm}, N_{\pm}] &= [Q_{\pm}, N_{\mp}] = [R, N_{\pm}] = [N_{\pm}, N_{\mp}] = [K_{\pm}, N_{\pm}] = [K_{\pm}, N_{\mp}] = 0, \\ \{Q_{\pm}, K_{\mp}\} &= \{K_+, K_-\} = Q_{\pm}^2 = K_{\pm}^2 = 0, \end{aligned} \quad (2.12)$$

This algebra (2.12) is useful to analyze a cohomology of nilpotent SUSY, because K_{\pm} plays an analogous role of homotopy operator.

Let us consider a monomial $\mathcal{O}_{\mathbf{k}}$ consists of the fundamental fields, where \mathbf{k} stands for a collection of lattice site of each field in the monomial, e.g. $\mathcal{O}_{n,m,l} = \chi_n \phi_m \phi_l$. And also consider a linear sum of a monomial over its lattice sites $\mathcal{O} = \sum_{\mathbf{k}} C_{\mathbf{k}} \mathcal{O}_{\mathbf{k}}$. If the coefficient $C_{\mathbf{k}}$ satisfies the following two conditions, then we say the $C_{\mathbf{k}}$ is *translationally invariant local coefficient* or simply TILC.

1. *Translational invariance:* $C_{\mathbf{k}}$ is invariant under discrete translation of lattice site, i.e. $C_{m_0+1, m_1+1, \dots, m_I+1} = C_{m_0, m_1, \dots, m_I}$. Then it is a function only of the site differences,

$$C_{m_0, m_1, \dots, m_I} = C(m_0 - m_1, m_0 - m_2, \dots, m_0 - m_I). \quad (2.13)$$

2. *Locality*: For each index (site difference) k_ℓ , if k_ℓ is large enough, then there exist $M, L > 0$ such that

$$|C(k_1, k_2, \dots, k_I)| < L \exp(-M|k_\ell|) \quad (2.14)$$

If C is TILC, then we can define holomorphic function

$$\tilde{C}(z_1, z_2, \dots, z_I) \equiv \sum_{\mathbf{k}} C(k_1, k_2, \dots, k_I) z_1^{k_1} z_2^{k_2} \dots z_I^{k_I} \quad (2.15)$$

which is holomorphic in an I -dimensional complex domain,

$$\mathcal{D}^I = \{1 - \epsilon_1 < z_1 < 1 + \epsilon_1, 1 - \epsilon_2 < z_2 < 1 + \epsilon_2, \dots, 1 - \epsilon_I < z_I < 1 + \epsilon_I | \epsilon_i > 0\}. \quad (2.16)$$

We call \tilde{C} holomorphic-representation or simply H-representation of C .

As for the fields, we introduce “superfields” which are slightly different from conventional ones in the continuum theory, but correspond to a certain rearrangement of the original fields. With the Grassmann variables θ_\pm , we define eight superfields Φ_\pm , Ψ_\pm , Υ_\pm and S_\pm :

$$\begin{aligned} \Phi_{\pm n} &\equiv \phi_{\pm n} \pm \theta_\pm \bar{\chi}_{\pm n}, & \Upsilon_{\pm n} &\equiv F_{\pm n} - i\theta_\mp (\nabla \bar{\chi}_\pm)_n, \\ \Psi_{\pm n} &\equiv \chi_{\pm n} + \theta_\pm \{F_{\pm n} - i\theta_\mp (\nabla \bar{\chi}_\pm)_n\} \pm i\theta_\mp (\nabla \phi_\pm)_n, & S_{\pm n} &\equiv \bar{\chi}_{\pm n}. \end{aligned} \quad (2.17)$$

Then nilpotent SUSY transformations δ_\pm have simple expressions on the superfields, namely derivatives with respect to θ_\pm :

$$\delta_\pm \Xi_n = \frac{\partial}{\partial \theta_\pm} \Xi_n, \quad \text{where} \quad \Xi_n = \Phi_{\pm n}, \Psi_{\pm n}, \Upsilon_{\pm n}, S_{\pm n}. \quad (2.18)$$

Figure 2 depicted diagrammatically these transformations of the superfields.

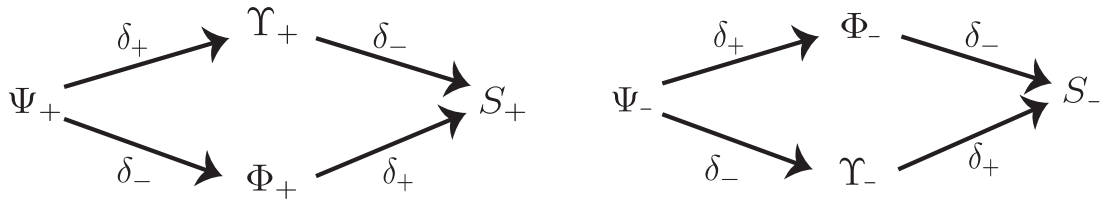


Figure 2: δ_\pm transformation for superfields

For any functional \mathcal{O} made of the above superfields, the nilpotent supersymmetry transformations act as

$$Q_\pm \mathcal{O} = \frac{\partial}{\partial \theta_\pm} \mathcal{O}. \quad (2.19)$$

Furthermore, any functional O of original fields (2.1) can be expressed in terms of that of superfields

$$O = \int d^2\theta \theta_+ \theta_- \mathcal{O} = \mathcal{O}|_{\theta_\pm=0}, \quad (2.20)$$

where we used the following integral formulae for the Grassmann variables θ_{\pm} ,

$$\int d^2\theta \equiv \int d\theta_- d\theta_+, \quad \int d\theta_+ \theta_+ = \int d\theta_- \theta_- = 1, \quad \int d^2\theta \theta_+ \theta_- = 1. \quad (2.21)$$

Note that our superfields are merely a change of variables from the original fields; the number of superfields is the same as that of the original fields, and the expression of any quantity in terms of original fields can be recovered from that of superfields by putting $\theta_{\pm} = 0$. Nonetheless, because of the simplicity of the expression (2.18) of Q_{\pm} , the relations (2.19) and (2.20) will largely simplify the classification of supersymmetric invariant local functionals in the fermion number zero sector in later section. Therefore, we shall use these eight superfields instead of eight fundamental fields for the discussion of the Q_{\pm} -cohomology and for a proof of nonrenormalization theorem. Quantum numbers for these superfields and θ_{\pm} are assigned as listed in Table 2.

Table 2: Quantum numbers for superfields and θ_{\pm}

	Φ_{\pm}	Ψ_{\pm}	Υ_{\pm}	S_{\pm}	θ_{\pm}
N_F	0	1	0	-1	1
$U(1)$	± 1	± 1	± 1	± 1	0
$U(1)_R$	± 1	0	∓ 1	0	± 1

2.2 Nilpotent-SUSY invariant action and cyclic Leibniz rule

Now, we write down a kinetic term S_0 of the nilpotent-SUSY invariant tree action,

$$\begin{aligned} S_0 &= \langle \nabla \phi_-, \nabla \phi_+ \rangle + i \langle \nabla \bar{\chi}_-, \chi_+ \rangle - i \langle \chi_-, \nabla \bar{\chi}_+ \rangle - \langle F_-, F_+ \rangle \\ &= \int d^2\theta \langle \Psi_-, \Psi_+ \rangle. \end{aligned} \quad (2.22)$$

Due to the translational invariance and locality, the difference operator ∇ in the above is actually a function of the site difference, i.e. $\nabla_{mn} = \nabla(m-n)$, and also is bounded for large $|k|$ as $|\nabla(k)| < L \exp(-M|k|)$ with positive real numbers L, M ; in other words, it is TILC.

In general, this kinetic term may have a doubling problem. In order to avoid it we introduce Wilson terms as well as ordinary mass terms,

$$\begin{aligned} S_m &= \langle F_+, G_+ \phi_+ \rangle - \langle \chi_+, G_+ \bar{\chi}_+ \rangle + \langle F_-, G_- \phi_- \rangle + \langle \chi_-, G_- \bar{\chi}_- \rangle \\ &= - \int d^2\theta \theta_- \langle \Psi_+, G_+ \Phi_+ \rangle + \int d^2\theta \theta_+ \langle \Psi_-, G_- \Phi_- \rangle. \end{aligned} \quad (2.23)$$

Here G_{\pm} expresses mass and Wilson terms for \pm fields. As a typical example, in the case of $(\nabla)_{mn} = (\delta_{m+1,n} - \delta_{m-1,n})/2$, we can take the term

$$\begin{aligned} (G_+)_{mn} &= m_+ \delta_{m,n} + r_+ (\delta_{m+1,n} - 2\delta_{m,n} + \delta_{m-1,n})/2, \\ (G_-)_{mn} &= m_- \delta_{m,n} + r_- (\delta_{m+1,n} - 2\delta_{m,n} + \delta_{m-1,n})/2, \end{aligned} \quad (2.24)$$

where m_{\pm} are mass parameters with $m_+^* = m_-$ and r_{\pm} are Wilson parameters with $r_+^* = r_-$. In the H-representation, they are expressed as

$$\tilde{\nabla}(w) = (w - w^{-1})/2, \quad \tilde{G}_{\pm}(w) = m_{\pm} + r_{\pm}(w - 2 + w^{-1})/2. \quad (2.25)$$

Note that, as far as site indices are symmetric, any choice of $(G_{\pm})_{mn} = (G_{\pm})_{nm}$ makes S_m invariant under nilpotent SUSY (2.2) with the symmetric difference operator $(\nabla)_{mn} = -(\nabla)_{nm}$.³

Let us turn to the interaction terms. We begin with the simplest case, namely three point interaction which can be written in the form

$$\begin{aligned} S_{int} &= \lambda_+ \langle F_+, \phi_+ * \phi_+ \rangle - \lambda_+ \langle \chi_+, \bar{\chi}_+ * \phi_+ \rangle - \lambda_+ \langle \chi_+, \phi_+ * \bar{\chi}_+ \rangle \\ &\quad + \lambda_- \langle F_-, \phi_- * \phi_- \rangle + \lambda_- \langle \chi_-, \bar{\chi}_- * \phi_- \rangle + \lambda_- \langle \chi_-, \phi_- * \bar{\chi}_- \rangle \\ &= - \int d^2\theta \theta_- \lambda_+ \langle \Psi_+, \Phi_+ * \Phi_+ \rangle + \int d^2\theta \theta_+ \lambda_- \langle \Psi_-, \Phi_- * \Phi_- \rangle \end{aligned} \quad (2.26)$$

where we denote a two-field product as

$$(A * B)_{\ell} \equiv \sum_{mn} M_{\ell mn} A_m B_n. \quad (2.27)$$

Here the coefficient $M_{\ell mn}$ is symmetric with respect to the last two indices, $M_{\ell mn} = M_{\ell nm}$, so that two bosonic fields are commutative and two fermionic fields are anti-commutative to each other. Three-point interaction action (2.26) is supersymmetric invariant if the product (2.27) satisfies the next relation for any bosonic fields A, B, C ,

$$\langle (\nabla A), B * C \rangle + \langle (\nabla B), C * A \rangle + \langle (\nabla C), A * B \rangle = 0, \quad (2.28)$$

$$\text{i.e.} \quad \sum_k (\nabla_{k\ell} M_{kmn} + \nabla_{km} M_{kn\ell} + \nabla_{kn} M_{k\ell m}) = 0. \quad (2.29)$$

This is the cyclic Leibniz rule (CLR) proposed in [7] and plays a crucial role in our formulation.

In order to extend to multi-point interaction more than three, we introduce multi-field symmetric product as an extension of (2.27),

$$[[B^{(1)}, B^{(2)}, \dots, B^{(k)}]]_m = \sum_{n_1 \dots n_k} M_{mn_1 \dots n_k} B_{n_1}^{(1)} \dots B_{n_k}^{(k)} \quad (2.30)$$

where the last k indices of the coefficient M are totally symmetric. The above $*$ -product corresponds to the $k = 2$ case: $(A * B)_m = [[A, B]]_m$. Again if this multi-field product satisfies the following CLR for any $k + 1$ bosonic fields $B^{(i)}$

$$\begin{aligned} \langle \nabla B^{(0)}, [[B^{(1)}, B^{(2)}, \dots, B^{(k)}]] \rangle &+ \langle \nabla B^{(1)}, [[B^{(2)}, B^{(3)}, \dots, B^{(0)}]] \rangle \\ &+ \dots + \langle \nabla B^{(k)}, [[B^{(0)}, B^{(1)}, \dots, B^{(k-1)}]] \rangle = 0, \end{aligned} \quad (2.31)$$

$$\text{i.e.} \quad \sum_{p: \text{cyclic perms of } \{0, 1, \dots, k\}} \sum_m \nabla_{mn_{p(0)}} M_{mn_{p(1)} \dots n_{p(k)}} = 0, \quad (2.32)$$

³Though confusing, difference operator with anti-symmetric site indices is usually called symmetric.

then a $(k+1)$ -point interaction term like

$$-\lambda_+^{(k+1)} \int d^2\theta \theta_- \langle \Psi_+, [\Phi_+, \dots, \Phi_+] \rangle + \lambda_-^{(k+1)} \int d^2\theta \theta_+ \langle \Psi_-, [\Phi_-, \dots, \Phi_-] \rangle \quad (2.33)$$

is invariant under nilpotent SUSY.

2.3 Effective action on lattice and Ward-Takahashi identities

We will discuss quantum corrections for our model in the subsequent sections. In order to make it clear of what kind of quantum symmetry our arguments treat, here we define the effective action and see its invariance under the supersymmetry. Let us begin by deriving the Ward-Takahashi identity for the 1-particle-irreducible(1PI) effective action $\Gamma(\phi)$ which is defined by the Legendre transformation $\Gamma(\phi) = \sum_{i,n} J_n^i \phi_n^i - W(J)$ from the generating functional $W(J)$ for the connected Green's functions. $W(J)$ is defined by the following path integral with source fields J_n^i

$$e^{W(J)} = \int \mathcal{D}\varphi e^{-S(\varphi) + \sum_{i,n} J_n^i \varphi_n^i}. \quad (2.34)$$

Here we use an index i for distinguishing various fields and n for lattice sites. The argument field ϕ_n^i of $\Gamma(\phi)$ is defined as an expectation value of φ_n^i under the source J :

$$\phi_n^i \equiv \frac{\partial W(J)}{\partial J_n^i} = \langle \varphi_n^i \rangle_J. \quad (2.35)$$

We use left-derivative convention for fermionic fields. So we have

$$J_n^i = (-1)^{|J^i||\phi^i|} \frac{\partial \Gamma(\phi)}{\partial \phi_n^i} \quad (2.36)$$

with grassmann parity $|A| = 0$ or 1 for grassmann even or odd field A , respectively.

Let us make the change of integration fields from φ to $\varphi + \delta\varphi$ in eq. (2.34). If we choose $\delta\varphi$ as an infinitesimal symmetry transformation and the action $S(\varphi)$ and the measure $\mathcal{D}\varphi$ are invariant under the transformation, then we obtain the identity

$$\int \mathcal{D}\varphi \sum_{i,n} J_n^i \delta\varphi_n^i e^{-S(\varphi) + \sum_{i,n} J_n^i \varphi_n^i} = 0, \quad (2.37)$$

that is,

$$\sum_{i,n} J_n^i \langle \delta\varphi_n^i \rangle_J = \sum_{i,n} \langle \delta\varphi_n^i \rangle_J \frac{\partial \Gamma(\phi)}{\partial \phi_n^i} = 0. \quad (2.38)$$

Since our supersymmetry transformations Q_\pm are linearly realized in the typical form of

$$\delta\varphi_n^i = \epsilon \sum_{j,m} A_{nm}^{ij} \varphi_m^j, \quad (2.39)$$

we find

$$\langle \delta\varphi_n^i \rangle_J = \epsilon \sum_{j,m} A_{nm}^{ij} \langle \varphi_m^j \rangle_J = \epsilon \sum_{j,m} A_{nm}^{ij} \phi_m^j. \quad (2.40)$$

Thus defining the transformation for ϕ_n^i as $\delta\phi_n^i = \epsilon \sum_{j,m} A_{nm}^{ij} \phi_m^j$, we obtain the Ward-Takahashi identity for the effective action

$$\delta\Gamma(\phi) \equiv \sum_{i,n} \delta\phi_n^i \frac{\partial\Gamma(\phi)}{\partial\phi_n^i} = 0. \quad (2.41)$$

This shows that the supersymmetry transformation for the effective action has the same form of a tree level action. This comes from the fact that we retain auxiliary fields and the transformation is kept linear. Also notice that the coefficients A_{nm}^{ij} in the transformation for the original field φ is inherited to those of ϕ , which means the difference operator in the coefficients is the same both for φ and ϕ . In other words, we can just replace original fields by its expectation values in supersymmetry transformations Q_\pm and the Ward-Takahashi identity becomes

$$Q_\pm \Gamma = 0. \quad (2.42)$$

In the following sections we use the same symbol for the original field and its expectation value in the discussion of the effective action for simplicity.

3. Cohomology of Q_\pm and the classification of Q_\pm -invariant functionals of fields

In this section, we are going to analyze cohomology of the nilpotent-SUSY transformations Q_\pm and classify all Q_\pm -invariant functionals of fields with TILC. As will be seen shortly, there are two categories which we call type-I and -II. Typical examples of the former are kinetic terms, while those of the latter are mass terms and interaction terms appeared in the tree action.

Since Q_\pm do not change the number of ‘+’-fields nor that of ‘-’-fields, we can safely restrict our discussion in the subsector with definite number of ‘ \pm ’-fields without loss of generality. If we write $\mathcal{O}(n_+, n_-)$ for the functional consisting of n_+ ‘+’-fields and n_- ‘-’-fields, then

$$N_\pm \mathcal{O}(n_+, n_-) = n_\pm \mathcal{O}(n_+, n_-), \quad (3.1)$$

for the operator defined in (2.11).

Immediate consequence at this point is the following. In use of the algebra $\{Q_\pm, K_\pm\} = N_\pm$ in (2.12), Q_\pm -closed functional $\mathcal{O}(n_+, n_-)$ can be written in the form

$$\mathcal{O}(n_+, n_-) = Q_+ K_+ \mathcal{O}(n_+, n_-) / n_+ \quad \text{if} \quad Q_+ \mathcal{O} = 0 \text{ and } n_+ \neq 0, \quad (3.2)$$

$$\mathcal{O}(n_+, n_-) = Q_- K_- \mathcal{O}(n_+, n_-) / n_- \quad \text{if} \quad Q_- \mathcal{O} = 0 \text{ and } n_- \neq 0. \quad (3.3)$$

Thus Q_+ and Q_- cohomologies in the space of functionals $\{\mathcal{O}(n_+, n_-)\}$ are trivial for $n_+ \neq 0$ and $n_- \neq 0$ respectively.

Our next task is to determine Q_- cohomology in the subspace $\{\mathcal{O}(n_+, 0)\}$ and Q_+ cohomology in the subspace $\{\mathcal{O}(0, n_-)\}$. Before stating the result, let us define the *CLR*

terms in the $N_F = 1$ sector as

$$\begin{aligned}\mathcal{C}_+^{N_F=1} &= \sum_{m, n_1, \dots, n_k} C_{mn_1 \dots n_k} \chi_{+m} \phi_{+n_1} \dots \phi_{+n_k} \quad \text{in } \{\mathcal{O}(k+1, 0)\}, \\ \mathcal{C}_-^{N_F=1} &= \sum_{m, n_1, \dots, n_k} C_{mn_1 \dots n_k} \chi_{-m} \phi_{-n_1} \dots \phi_{-n_k} \quad \text{in } \{\mathcal{O}(0, k+1)\}\end{aligned} \quad (3.4)$$

where the coefficients C are TILC and satisfy the CLR relation (2.31). Note that CLR terms with $N_F = 1$ have the maximum $U(1)_R$ charge (eigenvalue of the operator R) in the space $\{\mathcal{O}(k+1, 0)\}$ and the minimum in $\{\mathcal{O}(0, k+1)\}$.

Then we have the following theorem for $N_F = 1$ sector:

Theorem 1. (Fundamental theorem on the cohomology of nilpotent SUSY)

If \mathcal{P}_+ is a local functional in $\{\mathcal{O}(n_+, 0) | n_+ > 0\}$ with $N_F = 1$ and satisfies

$$Q_- \mathcal{P}_+ = 0, \quad (3.5)$$

then \mathcal{P}_+ can be written in the form

$$\mathcal{P}_+ = \mathcal{C}_+^{N_F=1} + Q_- \mathcal{Q}_+, \quad (3.6)$$

where \mathcal{Q}_+ is a local functional in $\{\mathcal{O}(n_+, 0) | n_+ > 0\}$ with $N_F = 2$.

Similarly, if \mathcal{P}_- is a local functional in $\{\mathcal{O}(0, n_-) | n_- > 0\}$ with $N_F = 1$ and satisfies

$$Q_+ \mathcal{P}_- = 0, \quad (3.7)$$

then \mathcal{P}_- can be written in the form

$$\mathcal{P}_- = \mathcal{C}_-^{N_F=1} + Q_+ \mathcal{Q}_-. \quad (3.8)$$

where \mathcal{Q}_- is a local functional in $\{\mathcal{O}(0, n_-) | n_- > 0\}$ with $N_F = 2$.

The proof for this theorem is shown in Appendix C with some preparations in Appendix B.

We turn to discuss a nilpotent-SUSY invariant local functional S with $N_F = 0$. The quantity shall be useful in discussing an effective action and proving a nonrenormalization theorem.

Proposition 1. A nilpotent-SUSY invariant local functional S with $N_F = 0$ can be generally written as

$$\begin{aligned}S &= Q_+ Q_- T(+, -) + Q_+ (\text{+type CLR terms with } N_F = 1) \\ &\quad + Q_- (\text{-type CLR terms with } N_F = 1),\end{aligned} \quad (3.9)$$

where $T(+, -)$ is a local functional with $N_F = 2$.

Proof. Since nilpotent SUSY-transformations Q_{\pm} do not change the number of $+$ -fields nor $-$ -fields, we can separately argue each term with definite numbers of \pm -fields. For a term with $n_+ \neq 0$ and $n_- \neq 0$ (let us denote \mathcal{T}), from (3.2) and (3.3) any nilpotent-SUSY invariant local functional takes the following form,

$$\mathcal{T} = Q_- Q_+ \left(K_+ K_- \mathcal{T} / (n_+ n_-) \right). \quad (3.10)$$

Note that both K_+ and K_- map from local functionals to local functionals.

For a term with $n_- = 0$, $n_+ \neq 0$ and $N_F = 0$ (let us denote \mathcal{U}_+), any nilpotent-SUSY invariant functional can be written as

$$\mathcal{U}_+ = Q_+ K_+ \mathcal{U}_+ / n_+. \quad (3.11)$$

From $Q_- \mathcal{U}_+ = 0$ and $\{Q_-, K_+\} = 0$ in (2.12), it follows that $\mathcal{P}_+ \equiv K_+ \mathcal{U}_+ / n_+$ is a Q_- -invariant local functional with $N_F = 1$. Thus we can apply Theorem 1 to \mathcal{P}_+ , so that we have

$$\mathcal{P}_+ = (+\text{type CLR terms with } N_F = 1) + Q_- \mathcal{Q}_+, \quad (3.12)$$

where \mathcal{Q}_+ is a local functional with $n_+ > 0$, $n_- = 0$ and $N_F = 2$. Thus

$$\mathcal{U}_+ = Q_+ \mathcal{P}_+ = Q_+ (+\text{type CLR terms with } N_F = 1) + Q_+ Q_- \mathcal{Q}_+, \quad (3.13)$$

From similar discussions, for a nilpotent-SUSY invariant local functional \mathcal{V}_- with $n_+ = 0$, $n_- \neq 0$ and $N_F = 0$, we can get

$$\mathcal{V}_- = Q_- (-\text{type CLR terms with } N_F = 1) + Q_- Q_+ \mathcal{Q}_-, \quad (3.14)$$

where \mathcal{Q}_- is a local functional with $n_+ = 0$, $n_- > 0$ and $N_F = 2$.

Thus, putting all terms together, we have

$$\begin{aligned} S &= Q_- Q_+ \left(K_+ K_- \mathcal{T} / (n_+ n_-) \right) + Q_+ (+\text{type CLR terms with } N_F = 1) + Q_+ Q_- \mathcal{Q}_+ \\ &\quad + Q_- (-\text{type CLR terms with } N_F = 1) + Q_- Q_+ \mathcal{Q}_- \\ &= Q_+ Q_- \left(-K_+ K_- \mathcal{T} / (n_+ n_-) + \mathcal{Q}_+ - \mathcal{Q}_- \right) \\ &\quad + Q_+ (+\text{type CLR terms with } N_F = 1) + Q_- (-\text{type CLR terms with } N_F = 1). \end{aligned} \quad (3.15)$$

This is indeed a form (3.9).⁴ □

The result (3.9) of Proposition 1 means that nilpotent-SUSY invariant local functionals with $N_F = 0$ are classified into (i) Q_+ and Q_- exact forms (type-I), (ii) Q_+ exact but not Q_- exact forms (type-II₊), and (iii) Q_- exact but not Q_+ exact forms (type-II₋). The important thing is that functionals of type-II are only CLR terms.

⁴Note that any local functional O with $n_+ \neq 0$ and $n_- \neq 0$ can be always expressed as $K_+ K_- O'$ by another local functional O' up to Q_{\pm} exact form functionals, since $1 = [Q_+ Q_-, K_+ K_-] / (n_+ n_-) + (Q_+ K_+ / n_+ + Q_- K_- / n_-)$ for $n_+ \neq 0$ and $n_- \neq 0$.

Let us consider a superfield counterpart of Proposition 1. From (2.19) and (2.20), we can translate a nilpotent SUSY transformed functional $Q_{\pm}O$ into a functional of superfields,

$$\begin{aligned} Q_{\pm}O &= \int d^2\theta\theta_+\theta_-Q_{\pm}\mathcal{O} \\ &= \int d^2\theta\theta_+\theta_-\frac{\partial}{\partial\theta_{\pm}}\mathcal{O} \\ &= \mp \int d^2\theta\theta_{\mp}\mathcal{O}. \end{aligned} \quad (3.16)$$

And also,

$$\begin{aligned} Q_+Q_-O &= - \int d^2\theta\theta_+\theta_-\frac{\partial}{\partial\theta_-}\frac{\partial}{\partial\theta_+}\mathcal{O} \\ &= - \int d^2\theta\mathcal{O}. \end{aligned} \quad (3.17)$$

By utilizing (3.16), (3.17) and (3.9), the next proposition follows:

Proposition 2. *A nilpotent-SUSY invariant local functional with $N_F = 0$ of superfields can be written in the form*

$$\begin{aligned} S &= \int d^2\theta\mathcal{T}(+,-) - \int d^2\theta\theta_-(+\text{type CLR terms with } N_F = 1) \\ &\quad + \int d^2\theta\theta_+(-\text{type CLR terms with } N_F = 1), \end{aligned} \quad (3.18)$$

where the “ \pm type CLR terms with $N_F = 1$ ” consisting of $k+1$ superfields ($k = 0, 1, 2, \dots$) are given by

$$\sum_{mn_1\cdots n_k} C_{mn_1\cdots n_k} \Psi_{\pm m} \Phi_{\pm n_1} \cdots \Phi_{\pm n_k}, \quad (3.19)$$

whose coefficient C is a TILC satisfying the CLR.

In Section 5, we will use a trick in which we lift constant parameters in the model, such as coupling constants, mass parameters and Wilson-term coefficients, to constant superfields by introducing constant super-partner for each constant parameter. Therefore we need an extended version of Proposition 2 including constant superfields. Let us take N_{CF} constant parameters ρ_{\pm}^i ($i = 1, \dots, N_{CF}$) as constant fields, and their super-partners $\zeta_{\rho\pm}^i$ ($i = 1, \dots, N_{CF}$) as well. Then $Q_{\pm}, K_{\pm}, N_{\pm}$ are modified to include constant fields:

$$Q'_{\pm} \equiv Q_{\pm} \pm \sum_i \zeta_{\rho\pm}^i \frac{\partial}{\partial\rho_{\pm}^i}, \quad K'_{\pm} \equiv K_{\pm} \pm \sum_i \rho_{\pm}^i \frac{\partial}{\partial\zeta_{\rho\pm}^i}, \quad N'_{\pm} \equiv N_{\pm} + \sum_i (\rho_{\pm}^i \frac{\partial}{\partial\rho_{\pm}^i} + \zeta_{\rho\pm}^i \frac{\partial}{\partial\zeta_{\rho\pm}^i}), \quad (3.20)$$

and their algebra is the same as before:

$$\begin{aligned} \{Q'_{\pm}, K'_{\pm}\} &= N'_{\pm}, \\ [Q'_{\pm}, N'_{\pm}] &= [Q'_{\pm}, N'_{\mp}] = [N'_{\pm}, N'_{\mp}] = [K'_{\pm}, N'_{\pm}] = [K'_{\pm}, N'_{\mp}] = 0, \\ \{Q'_{\pm}, K'_{\mp}\} &= \{K'_{\pm}, K'_{\mp}\} = Q'^2_{\pm} = K'^2_{\pm} = 0. \end{aligned} \quad (3.21)$$

We assign $N_F = 0$ for ρ_\pm^i and -1 for $\zeta_{\rho\pm}^i$. Then, \pm type local functionals are defined by new operators as

$$N'_\mp \mathcal{O}_\pm = 0, \quad N'_\pm \mathcal{O}_\pm \neq 0. \quad (3.22)$$

We should notice that the constant fields do not depend on lattice site, so the coefficient appeared in functionals only depend on the site index of ordinary fields to which the notion of locality refers.

From (3.20), we can easily show an extended version of the fundamental theorem as follows,

Theorem 2. *Let \mathcal{P}'_\pm be any \pm type local functional with $N_F = 1$. If*

$$Q'_\mp \mathcal{P}'_\pm = 0, \quad (3.23)$$

then

$$\mathcal{P}'_\pm = (\pm\text{type CLR extended terms with } N_F = 1) + Q'_\mp \mathcal{Q}'_\pm, \quad (3.24)$$

where \pm type CLR extended terms with $N_F = 1$ of $k+1$ -th order are expressed as

$$\sum_{mn_1 \cdots n_k} C'_{mn_1 \cdots n_k} \chi_{\pm m} \phi_{\pm n_1} \cdots \phi_{\pm n_k}, \quad (3.25)$$

and the coefficient C' is a TILC satisfying the CLR condition and may depend on constant fields ρ_\pm . \mathcal{Q}'_\pm are $N_F = 2$ local functionals which may depend on $\rho_\pm^i, \zeta_{\rho\pm}^i$.

The proof goes almost same way as that of the previous Theorem 1 with a care of the fact that $\rho_\pm^i, \zeta_{\rho\pm}^i$ are invariant under δ_\mp . We also note that $\zeta_{\rho\pm}^i$ does not contribute to nontrivial CLR terms with $N_F = 1$, like $\bar{\chi}_\pm$.

From this theorem, we can prove two additional propositions:

Proposition 3. *A nilpotent-SUSY invariant local functional S' with $N_F = 0$ is generally written in the form*

$$\begin{aligned} S' = & Q'_+ Q'_- T'(+, -) + Q'_+ (\text{+type CLR extended terms with } N_F = 1) \\ & + Q'_- (\text{-type CLR extended terms with } N_F = 1), \end{aligned} \quad (3.26)$$

where $T'(+, -)$ is a local functional with $N_F = 2$.

By introducing two kinds of constant superfields $\rho_\pm^i(\theta_\pm) \equiv \rho_\pm^i \pm \theta_\pm \zeta_{\rho\pm}^i$ and $\zeta_{\rho\pm}^i(\theta_\pm) \equiv \zeta_{\rho\pm}^i$, the action of Q'_\pm on any superfield functional \mathcal{O}' is expressed simply as

$$Q'_\pm \mathcal{O}' = \frac{\partial}{\partial \theta_\pm} \mathcal{O}'. \quad (3.27)$$

Similarly to (2.20), this expression (3.27) enables us to replace component fields in Proposition 3 to superfields.

Proposition 4. *A nilpotent-SUSY invariant $N_F = 0$ local functional of superfields can be written as*

$$S' = \int d^2\theta \mathcal{T}'(+, -) - \int d^2\theta \theta_- (+\text{type CLR extended terms with } N_F = 1) \\ + \int d^2\theta \theta_+ (-\text{type CLR extended terms with } N_F = 1), \quad (3.28)$$

where the above \pm type CLR extended terms with $N_F = 1$ of $k + 1$ -th order is defined as

$$\sum_{mn_1 \cdots n_k} C'_{mn_1 \cdots n_k}(\rho_{\pm}^i(\theta_{\pm})) \Psi_{\pm m} \Phi_{\pm n_1} \cdots \Phi_{\pm n_k}, \quad (3.29)$$

and the coefficient C' is a TILC satisfying the CLR condition and may depend on constant superfields $\rho_{\pm}^i(\theta_{\pm})$.

In summary of this section, all nilpotent-SUSY invariant local functionals with $N_F = 0$ are classified into type-I and type-II. The latter contains only a linear combination of CLR terms (3.4), (3.19), (3.25) and (3.29) as a consequence of Q_{\mp} -cohomology. It is a surprising result that there is no nilpotent-SUSY invariant type-II local functional with $N_F = 1$ except CLR terms. For other sectors, we can find nontrivial cohomology elements in $N_F < 1$ sector due to the existence of S_{\pm} and Φ_{\pm} , while in $N_F > 1$ sector there is no nilpotent-SUSY invariant type-II local functional.

4. Quantum effects for the model

Before discussing non-renormalization theorem in the subsequent section, let us look at the quantum correction explicitly, say, in one-loop level. The kinetic term, which is type-I functional, indeed get contributions from one-loop diagrams. On the other hand, the terms given by the type-II functionals such as mass terms and the interaction terms have no contributions from one-loop diagrams. Even for two-loop level there is no quantum correction to the type-II terms due to the CLR.

Let us recapitulate the action for the kinetic term S_0 and the mass term S_m defined in subsection 2.2:

$$S_0 = \langle \nabla \phi_-, \nabla \phi_+ \rangle + i \langle \nabla \bar{\chi}_-, \chi_+ \rangle - i \langle \chi_-, \nabla \bar{\chi}_+ \rangle - \langle F_-, F_+ \rangle, \\ = \int d^2\theta \langle \Psi_-, \Psi_+ \rangle, \quad (4.1)$$

$$S_m = \langle F_+, G_+ \phi_+ \rangle - \langle \chi_+, G_+ \bar{\chi}_+ \rangle + \langle F_-, G_- \phi_- \rangle + \langle \chi_-, G_- \bar{\chi}_- \rangle \\ = - \int d^2\theta \theta_- \langle \Psi_+, G_+ \Phi_+ \rangle + \int d^2\theta \theta_+ \langle \Psi_-, G_- \Phi_- \rangle, \quad (4.2)$$

where G_{\pm} include Wilson terms (2.24),(2.25).

We could think the mass term as one of the interaction terms and might pursuit our perturbative calculation with massless propagator. In order to avoid infrared divergences, instead, we use perturbation with massive propagator. We denote a symbol $\langle \cdots \rangle_0$ as an

expectation value in the tree-level. By defining $D_{mn} = D_{nm} \equiv (\nabla^T \nabla + G_- G_+)^{-1}_{mn}$, the tree-level propagators can be written as

$$\begin{aligned}\langle \phi_{\mp m} \phi_{\pm n} \rangle_0 &= D_{mn}, & \langle \phi_{\pm m} \phi_{\pm n} \rangle_0 &= 0, \\ \langle \chi_{\mp m} \bar{\chi}_{\pm n} \rangle_0 &= -i(\nabla D)_{mn}, & \langle \chi_{\pm m} \bar{\chi}_{\pm n} \rangle_0 &= \pm(G_{\mp} D)_{mn}, \\ \langle F_{\mp m} F_{\pm n} \rangle_0 &= -(\nabla D \nabla^T)_{mn}, & \langle F_{\pm m} \phi_{\pm n} \rangle_0 &= (G_{\mp} D)_{mn},\end{aligned}\quad (4.3)$$

where $(\nabla^T)_{nm} = (\nabla)_{mn}$. Note that our mass operators $G_+, G_- = (G_+)^{\dagger}$ include Wilson terms, thus have site-dependence in general. In terms of superfields, relevant 2-point functions are

$$\begin{aligned}\langle \Phi_{\mp m}(\theta) \Phi_{\pm n}(\theta') \rangle_0 &= D_{mn}, & \langle \Phi_{\pm m}(\theta) \Phi_{\pm n}(\theta') \rangle_0 &= 0, \\ \langle \Psi_{\mp m}(\theta) \Phi_{\pm n}(\theta') \rangle_0 &= \mp i \delta(\theta_{\pm} - \theta'_{\pm}) (\nabla D)_{mn}, \\ \langle \Psi_{\pm m}(\theta) \Phi_{\pm n}(\theta') \rangle_0 &= \delta(\theta_{\pm} - \theta'_{\pm}) (G_{\mp} D)_{mn}, \\ \langle \Psi_{\mp m}(\theta) \Psi_{\pm n}(\theta') \rangle_0 &= i \delta^2(\theta - \theta') (\nabla D \nabla^T)_{mn}, \\ \langle \Psi_{\pm m}(\theta) \Psi_{\pm n}(\theta') \rangle_0 &= i \delta^2(\theta - \theta') (\nabla D G_{\mp}^T)_{mn} = -i \delta^2(\theta - \theta') (G_{\mp} D \nabla^T)_{mn},\end{aligned}\quad (4.4)$$

where δ -functions for the Grassmann variables are defined as $\delta(\theta_{\pm}) \equiv \theta_{\pm}$, $\delta^2(\theta) \equiv \theta_+ \theta_-$.

For the interaction term, we only consider three-point interaction for simplicity:

$$\begin{aligned}S_{int} &= \lambda_+ (\langle F_+, \phi_+ * \phi_+ \rangle - 2 \langle \chi_+, \bar{\chi}_+ * \phi_+ \rangle) \\ &\quad + \lambda_- (\langle F_-, \phi_- * \phi_- \rangle + 2 \langle \chi_-, \bar{\chi}_- * \phi_- \rangle) \\ &= -\lambda_+ \int d^2 \theta \theta_- \langle \Psi_+, \Phi_+ * \Phi_+ \rangle + \lambda_- \int d^2 \theta \theta_+ \langle \Psi_-, \Phi_- * \Phi_- \rangle,\end{aligned}\quad (4.5)$$

where $\lambda_{\mp} = \lambda_{\pm}^*$ and the coefficient M in the definition of $*$ product (2.27) is a TILC satisfying the CLR condition,

$$\sum_k (\nabla_{k\ell} M_{kmn} + \nabla_{km} M_{kn\ell} + \nabla_{kn} M_{k\ell m}) = \sum_k (\nabla_{\ell k}^T M_{kmn} + \nabla_{mk}^T M_{kn\ell} + \nabla_{nk}^T M_{k\ell m}) = 0. \quad (4.6)$$

4.1 Corrections for kinetic terms

One-loop corrections to the kinetic term $F_- F_+$, for instance, are given by Figure 3. Here internal lines with filled circles symbolize massive propagators.

At the one-loop level, we can write down the $F_- F_+$ propagator with correction in the form

$$\begin{aligned}\langle F_{-m} F_{+n} \rangle &= \langle F_{-m} F_{+n} \rangle_0 + \langle F_{-m} F_{+n'} \rangle_0 \Sigma_{FF}^{+-}(n', m') \langle F_{-m'} F_{+n} \rangle_0 \\ &\quad + \langle F_{-m} \phi_{-n'} \rangle_0 \Sigma_{\phi\phi}^{--}(n', m') \langle \phi_{+m'} F_{+n} \rangle_0\end{aligned}\quad (4.7)$$

and the self-energies at the one-loop are explicitly given by

$$\Sigma_{FF}^{+-}(n, m) = 2\lambda_+ \lambda_- M_{nk\ell} M_{mij} D_{ik} D_{j\ell}, \quad (4.8)$$

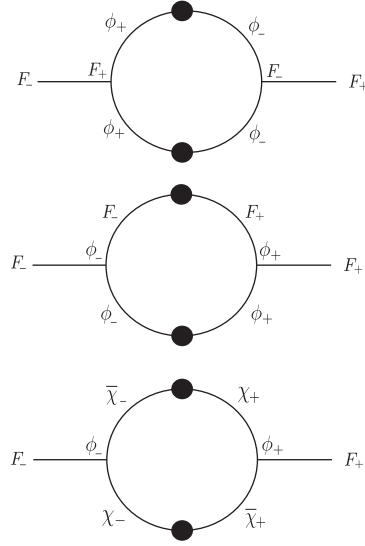


Figure 3: One-loop corrections for $F_- F_+$.

and

$$\begin{aligned}
\Sigma_{\phi\phi}^{-+}(n, m) &= -4\lambda_+\lambda_- M_{\ell kn} M_{ijm} (\nabla D \nabla^T)_{\ell i} D_{kj} \\
&\quad -4\lambda_+\lambda_- M_{k\ell n} M_{ijm} (\nabla D)_{kj} (D \nabla^T)_{\ell i} \\
&= 4\lambda_+\lambda_- (M \nabla)_{n k \ell} (\nabla^T M)_{ijm} D_{\ell i} D_{kj} \\
&= -2\lambda_+\lambda_- (M \nabla)_{n k \ell} (\nabla^T M)_{mij} D_{j\ell} D_{ik} \\
&= -(\nabla^T \Sigma_{FF}^{+-} \nabla)(n, m),
\end{aligned} \tag{4.9}$$

where we have used the symmetric property of $D_{mn} = D_{nm}$ in (4.3) and the CLR relation (4.6). Short-hand notations $\nabla_{\ell k} M_{\ell mn} \equiv (\nabla^T M)_{kmn}$ and $M_{k\ell m} \nabla_{mn} = M_{k\ell m} \nabla_{mn} \equiv (M \nabla)_{k\ell n}$ are also used.

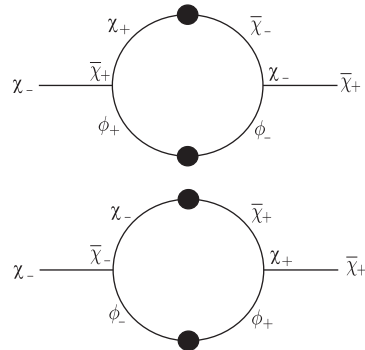


Figure 4: One-loop corrections for $\chi_- \bar{\chi}_+$.

The $\chi_- \bar{\chi}_+$ propagator is evaluated at the one-loop level as

$$\begin{aligned} \langle \chi_- \bar{\chi}_+ \rangle &= \langle \chi_- \bar{\chi}_+ \rangle_0 + \langle \chi_- \bar{\chi}_+ \rangle_0 \Sigma_{\bar{\chi}\chi}^{+-}(n', m') \langle \chi_- \bar{\chi}_+ \rangle_0 \\ &\quad + \langle \chi_- \bar{\chi}_- \rangle_0 \Sigma_{\bar{\chi}\chi}^{-+}(n', m') \langle \chi_+ \bar{\chi}_+ \rangle_0, \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} \Sigma_{\bar{\chi}\chi}^{+-}(n, m) &= 4i\lambda_+ \lambda_- M_{kn\ell} M_{mij} (D\nabla)_{ik} D_{j\ell} \\ &= -2i\lambda_+ \lambda_- (M\nabla)_{nkl} M_{mij} D_{ik} D_{j\ell} \\ &= -i(\nabla^T \Sigma_{FF}^{+-})(n, m). \end{aligned} \quad (4.11)$$

In this calculation the CLR (4.6) is important. Similarly,

$$\begin{aligned} \Sigma_{\bar{\chi}\chi}^{-+}(n, m) &= 4i\lambda_+ \lambda_- M_{\ell kn} M_{mij} (\nabla D)_{\ell j} D_{ki} \\ &= -i(\nabla^T \Sigma_{FF}^{+-})(n, m) = \Sigma_{\bar{\chi}\chi}^{+-}(n, m). \end{aligned} \quad (4.12)$$

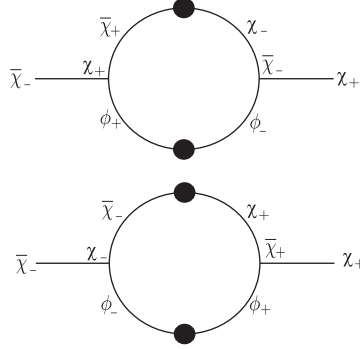


Figure 5: One-loop corrections for $\bar{\chi}_- \chi_+$.

For $\bar{\chi}_- \chi_+$ at the one-loop level, the propagator is evaluated as

$$\begin{aligned} \langle \bar{\chi}_- \chi_+ \rangle &= \langle \bar{\chi}_- \chi_+ \rangle_0 + \langle \bar{\chi}_- \chi_+ \rangle_0 \Sigma_{\bar{\chi}\chi}^{+-}(n', m') \langle \bar{\chi}_- \chi_+ \rangle_0 \\ &\quad + \langle \bar{\chi}_- \chi_- \rangle_0 \Sigma_{\bar{\chi}\chi}^{-+}(n', m') \langle \chi_+ \bar{\chi}_+ \rangle_0, \end{aligned} \quad (4.13)$$

where

$$\Sigma_{\bar{\chi}\chi}^{+-}(n, m) = 4i\lambda_+ \lambda_- M_{nkl} M_{imj} (\nabla D)_{ik} D_{j\ell}, \quad (4.14)$$

and

$$\Sigma_{\bar{\chi}\chi}^{-+}(n, m) = 4i\lambda_+ \lambda_- M_{nkl} M_{jim} (\nabla D)_{j\ell} D_{ki}. \quad (4.15)$$

In use of the CLR (4.6), self-energy parts (4.8), (4.14) and (4.15) are related as

$$\Sigma_{\bar{\chi}\chi}^{-+}(n, m) = \Sigma_{\bar{\chi}\chi}^{+-}(n, m) = -i(\Sigma_{FF}^{+-} \nabla)(n, m). \quad (4.16)$$

Finally, the $\phi_- \phi_+$ propagator at the one-loop level becomes

$$\begin{aligned} \langle \phi_- \phi_+ \rangle &= \langle \phi_- \phi_+ \rangle_0 + \langle \phi_- \phi_+ \rangle_0 \Sigma_{\phi\phi}^{+-}(n', m') \langle \phi_- \phi_+ \rangle_0 \\ &\quad + \langle \phi_- \phi_- \rangle_0 \Sigma_{\phi\phi}^{-+}(n', m') \langle \phi_+ \phi_+ \rangle_0. \end{aligned} \quad (4.17)$$

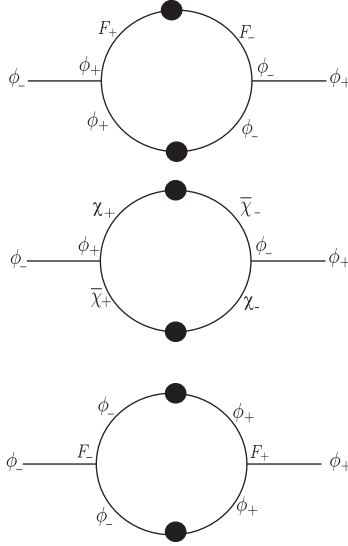


Figure 6: One-loop corrections for $\phi_- \phi_+$.

In summary, we have evaluated one-loop contributions to the part of self-energy which mixes \pm -fields and thus to the type-I functionals, i.e. kinetic terms. Relevant diagrams are shown in Figures 3, 4, 5 and 6. Note that

$$\Sigma_{FF}^{+-}(n, m) = \Sigma_{FF}^{-+}(m, n), \quad \Sigma_{\phi\phi}^{+-}(n, m) = \Sigma_{\phi\phi}^{-+}(m, n). \quad (4.18)$$

Also we can see each self-energy is related to each other by using the CLR (4.6) repeatedly,

$$(\nabla^T \Sigma_{FF}^{+-})(n, m) = (\nabla^T \Sigma_{FF}^{-+})(n, m) = i(\Sigma_{\chi\chi}^{\pm\mp})(n, m), \quad (4.19)$$

$$(\Sigma_{FF}^{+-} \nabla)(n, m) = (\Sigma_{FF}^{-+} \nabla)(n, m) = i(\Sigma_{\chi\chi}^{\pm\mp})(n, m), \quad (4.20)$$

$$\Sigma_{\phi\phi}^{+-}(n, m) = \Sigma_{\phi\phi}^{-+}(n, m) = -i(\Sigma_{\chi\chi}^{+-} \nabla)(n, m), \quad (4.21)$$

$$\Sigma_{\phi\phi}^{+-}(n, m) = \Sigma_{\phi\phi}^{-+}(n, m) = -i(\Sigma_{\chi\chi}^{+-} \nabla^T)(m, n). \quad (4.22)$$

These equations are exact relations thanks to the CLR and are nothing but a part of the SUSY Ward-Takahashi identities.

4.2 Quantum effects on type-II functionals

Let us turn to the type-II functionals, the simplest of which is the mass term. So we have to evaluate self-energy parts which do not mix \pm -fields. Immediate consequences we obtain are the vanishing of $\Sigma_{F\phi}^{++}$ and $\Sigma_{\chi\bar{\chi}}^{++}$ due to the vanishing of $\langle \phi_+ \phi_+ \rangle_0$ from (4.3). (We can easily see from Figure 7 that these self-energy parts are proportional to $\langle \phi_+ \phi_+ \rangle_0$.)

We emphasize here that not only pure mass term but also Wilson term have no quantum correction at one-loop (and actually any loop order as will be shown in next section), thus the Wilson parameter is unchanged.

The situation becomes a bit nontrivial in two-loop order. Actually, for the diagrams

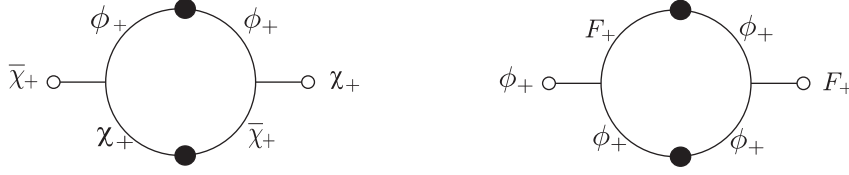


Figure 7: Diagrams contributing to $\Sigma_{\chi\chi}^{++}$ and $\Sigma_{F\phi}^{++}$ where open circle stands for amputated propagator. Both diagrams are proportional to $\langle\phi_+\phi_+\rangle_0$.

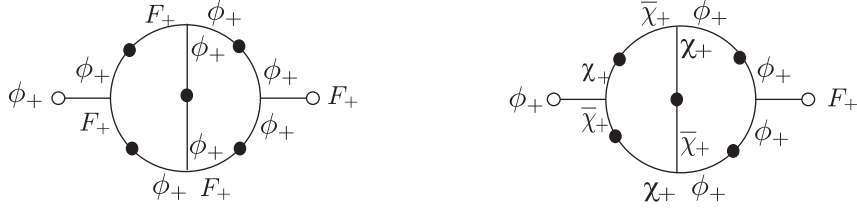


Figure 8: $O(\lambda_+^4)$ contribution to $\Sigma_{F\phi}^{++}$ at the 2-loop level. These graphs are proportional to $(\langle\phi_+\phi_+\rangle_0)^2$.

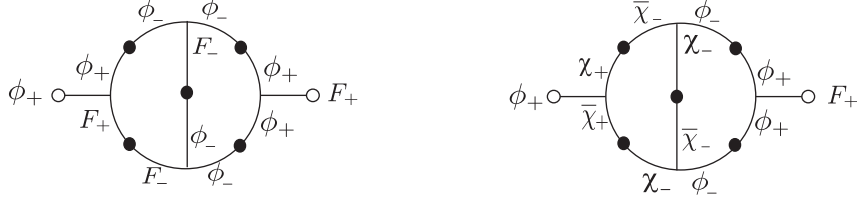


Figure 9: $O(\lambda_+^2 \lambda_-^2)$ contribution to $\Sigma_{F\phi}^{++}$ at 2-loop level.

in Figure 8, the correction for $F_+\phi_+$ is proportional to $\lambda_+^4 \langle\phi_+\phi_+\rangle_0^2$ and thus it vanishes as before. On the other hand, diagrams in Figure 9 with $\lambda_+^2 \lambda_-^2$ can be shown to be proportional to the difference operator ∇ by the CLR (4.6),

$$\begin{aligned}
\Sigma_{F\phi}^{++}(n, m)|_{2\text{-loop}} &= -2^4(\lambda_+\lambda_-)^2 \left((\nabla^T M)_{inj} + (\nabla^T M)_{jni} \right) M_{mkl} \bar{M}_{abc} (\bar{M} \nabla)_{def} \\
&\quad \times D_{bj} D_{ck} D_{f\ell} D_{di} D_{ae}^{-F\phi} \\
&= 2^4(\lambda_+\lambda_-)^2 (\nabla^T M)_{nij} M_{mkl} \bar{M}_{abc} (\bar{M} \nabla)_{def} D_{bj} D_{ck} D_{f\ell} D_{di} D_{ae}^{-F\phi},
\end{aligned} \tag{4.23}$$

where $D_{ij}^{\pm F\phi} \equiv \langle F_{\pm i} \phi_{\pm j} \rangle_0$. Since n is the site index of external ϕ_+ , the correction to the effective action is proportional to $\nabla \phi_+$. Therefore, this effect can be shown to be a 2-loop contribution to type-I functional.

5. Non-renormalization theorem on the lattice

5.1 Perturbative proof of the nonrenormalization theorem for type-II functionals

In the previous section, we explicitly saw that there is no quantum correction in mass and Wilson terms at the one loop level, while the kinetic terms have nontrivial quantum corrections. In this section, we show the non-renormalization theorem that the type-II terms are not suffered from quantum corrections at all.

Our starting action in the tree level is

$$S = S_{\text{type-I}} + S_{\text{type-II}}, \quad (5.1)$$

where

$$S_{\text{type-I}} = \int d\theta_- d\theta_+ \langle \Psi_-, \Psi_+ \rangle, \quad (5.2)$$

$$S_{\text{type-II}} = \int d\theta_- d\theta_+ \left\{ \theta_- \langle \Psi_+, W_+^{\text{tree}}(\Phi_+; v_+, m_+, \lambda_+) \rangle \right. \\ \left. - \theta_+ \langle \Psi_-, W_-^{\text{tree}}(\Phi_-; v_-, m_-, \lambda_-) \rangle \right\} \quad (5.3)$$

with

$$W_{\pm}^{\text{tree}}(\Phi_{\pm}; v_{\pm}, m_{\pm}, \lambda_{\pm}) = -v_{\pm} - m_{\pm} G_{\pm} \Phi_{\pm} - \lambda_{\pm} \Phi_{\pm} * \Phi_{\pm}. \quad (5.4)$$

Here we have a little bit generalized from the preceding sections by adding a linear term $v_{\pm} \Psi_{\pm}$ in the action as a type-II functional. The inclusion of the linear term is not a necessary thing but the argument below goes well either with or without this term. Also we have slightly changed our notation as $G_{\pm} \rightarrow m_{\pm} G_{\pm}$ from (2.24), and so this is accompanied by the change of the Wilson coefficient $r_{\pm} \rightarrow r_{\pm}/m_{\pm}$.

Type-II part of the effective action with full quantum corrections can be separated into the sum of tree action $S_{\text{type-II}}$ and quantum correction $\Delta\Gamma_{\text{type-II}}^{\text{eff}}$

$$\Gamma_{\text{type-II}}^{\text{eff}} = S_{\text{type-II}} + \Delta\Gamma_{\text{type-II}}^{\text{eff}}. \quad (5.5)$$

First of all, we are going to show that the latter should be of the form

$$\Delta\Gamma_{\text{type-II}}^{\text{eff}} = \int d\theta_- d\theta_+ \left\{ \theta_- \langle \Psi_+, W_+^{\text{eff}}(\Phi_+; v_+, m_+, \lambda_+) \rangle \right. \\ \left. - \theta_+ \langle \Psi_-, W_-^{\text{eff}}(\Phi_-; v_-, m_-, \lambda_-) \rangle \right\}. \quad (5.6)$$

Here emphasis is on that W_+^{eff} (W_-^{eff}) depends only on $\Phi_+, v_+, m_+, \lambda_+$ ($\Phi_-, v_-, m_-, \lambda_-$) but not on $\Phi_-, v_-, m_-, \lambda_-$ ($\Phi_+, v_+, m_+, \lambda_+$), and this holomorphic property (even in the parameter dependence) will turn out to be crucial in the proof of the non-renormalization theorem.

	Ψ_{\pm}	Φ_{\pm}	v_{\pm}	m_{\pm}	λ_{\pm}	θ_{\pm}	$d\theta_{\pm}$	S	$\Delta\Gamma_{\text{type II}}^{\text{eff}}$	$\frac{v_{\pm}\lambda_{\pm}}{(m_{\pm})^2}$	$\frac{\lambda_{\pm}\Phi_{\pm}}{m_{\pm}}$	$\frac{v_{\pm}}{m_{\pm}\Phi_{\pm}}$
N_F	+1	0	0	0	0	+1	-1	0	0	0	0	0
$U(1)$	± 1	± 1	∓ 1	∓ 2	∓ 3	0	0	0	0	0	0	0
$U(1)_R$	0	± 1	± 1	0	∓ 1	± 1	∓ 1	0	0	0	0	0

Table 3: the fermion number N_F , $U(1)$ and $U(1)_R$ charges

To this end, we replace the parameters v_{\pm} , m_{\pm} and λ_{\pm} by *constant superfields* such a way as [2]⁵

$$\begin{aligned}
v_{\pm} &\longrightarrow v_{\pm}(\theta_{\pm}) = v_{\pm} + \theta_{\pm}\zeta_{\pm}^v, \\
m_{\pm} &\longrightarrow m_{\pm}(\theta_{\pm}) = m_{\pm} + \theta_{\pm}\zeta_{\pm}^m, \\
\lambda_{\pm} &\longrightarrow \lambda_{\pm}(\theta_{\pm}) = \lambda_{\pm} + \theta_{\pm}\zeta_{\pm}^{\lambda},
\end{aligned} \tag{5.7}$$

where the lowest components of $v_{\pm}(\theta_{\pm})$, $m_{\pm}(\theta_{\pm})$ and $\lambda_{\pm}(\theta_{\pm})$ correspond to the original parameters v_{\pm} , m_{\pm} and λ_{\pm} . Note that the tree action (5.1) is still supersymmetric under the replacement (5.7) with the supersymmetry transformations:

$$\begin{aligned}
Q_{\pm}v_{\pm}(\theta_{\pm}) &= \frac{\partial}{\partial\theta_{\pm}}v_{\pm}(\theta_{\pm}), & \delta_{\mp}v_{\pm}(\theta_{\pm}) &= 0, \\
Q_{\pm}m_{\pm}(\theta_{\pm}) &= \frac{\partial}{\partial\theta_{\pm}}m_{\pm}(\theta_{\pm}), & \delta_{\mp}m_{\pm}(\theta_{\pm}) &= 0, \\
Q_{\pm}\lambda_{\pm}(\theta_{\pm}) &= \frac{\partial}{\partial\theta_{\pm}}\lambda_{\pm}(\theta_{\pm}), & \delta_{\mp}\lambda_{\pm}(\theta_{\pm}) &= 0.
\end{aligned} \tag{5.8}$$

Then, it is straightforward application of Proposition 4 to show that $\Delta\Gamma_{\text{type-II}}^{\text{eff}}$ can be supersymmetric only if W_+^{eff} (W_-^{eff}) is independent of $\Phi_-(\theta_-)$, $v_-(\theta_-)$, $m_-(\theta_-)$, $\lambda_-(\theta_-)$ ($\Phi_+(\theta_+)$, $v_+(\theta_+)$, $m_+(\theta_+)$, $\lambda_+(\theta_+)$).

As a next step, we further restrict the form of the functions $W_{\pm}^{\text{eff}}(\Phi_{\pm}; v_{\pm}, m_{\pm}, \lambda_{\pm})$ by assigning the fermion number N_F , the $U(1)$ and $U(1)_R$ charges to the fields, as listed in Table 3. Then, $\Delta\Gamma_{\text{type-II}}^{\text{eff}}$ is found to be generally expressed as

$$\begin{aligned}
\Delta\Gamma_{\text{type-II}}^{\text{eff}}[\Psi_{\pm}, \Phi_{\pm}; v_{\pm}, m_{\pm}, \lambda_{\pm}] &= \int d\theta_- d\theta_+ \left\{ \theta_- \langle \Psi_+, \frac{(m_+)^2}{\lambda_+} f_+ \left(\frac{v_+\lambda_+}{(m_+)^2}, \frac{\lambda_+\Phi_+}{m_+} \right) \right. \\
&\quad \left. - \theta_+ \langle \Psi_-, \frac{(m_-)^2}{\lambda_-} f_- \left(\frac{v_-\lambda_-}{(m_-)^2}, \frac{\lambda_-\Phi_-}{m_-} \right) \rangle \right\}, \tag{5.9}
\end{aligned}$$

where $f_{\pm}(z_{\pm}, w_{\pm})$ are some holomorphic functions of the complex variables $z_{\pm} = \frac{v_{\pm}\lambda_{\pm}}{(m_{\pm})^2}$ and $w_{\pm} = \frac{\lambda_{\pm}\Phi_{\pm}}{m_{\pm}}$.

⁵In Ref.[2], the mass parameters and the coupling constants have been replaced by chiral superfields having the space-time coordinate-dependence. Here, we assume that $v_{\pm}(\theta_{\pm})$, $m_{\pm}(\theta_{\pm})$ and $\lambda_{\pm}(\theta_{\pm})$ are the functions of the Grassmann coordinates θ_{\pm} but do not have the site-dependence of the lattice. This is because the SUSY invariance on the lattice will be lost if they depend on the site.

In this perturbative calculation, we may expand the functions $f_{\pm}(z_{\pm}, w_{\pm})$ in powers of z_{\pm} and w_{\pm} as

$$\begin{aligned} \Delta\Gamma_{\text{type-II}}^{\text{eff}}[\Psi_{\pm}, \Phi_{\pm}; v_{\pm}, m_{\pm}, \lambda_{\pm}] \\ = \int d\theta_- d\theta_+ \sum_k \sum_l \left\{ \theta_- \frac{(m_+)^2}{\lambda_+} a_{kl}^+ \left(\frac{v_+ \lambda_+}{(m_+)^2} \right)^k \left(\frac{\lambda_+}{m_+} \right)^l \langle \Psi_+, \underbrace{[\Phi_+, \Phi_+, \dots, \Phi_+]}_l \rangle \right. \\ \left. - \theta_+ \frac{(m_-)^2}{\lambda_-} a_{kl}^- \left(\frac{v_- \lambda_-}{(m_-)^2} \right)^k \left(\frac{\lambda_-}{m_-} \right)^l \langle \Psi_-, \underbrace{[\Phi_-, \Phi_-, \dots, \Phi_-]}_l \rangle \right\}, \quad (5.10) \end{aligned}$$

where a_{kl}^{\pm} are some constant coefficients. The SUSY invariance then requires the CLR relations

$$\langle \nabla \Phi_{\pm}, [\Phi_{\pm}, \Phi_{\pm}, \dots, \Phi_{\pm}] \rangle = 0, \quad (5.11)$$

as they should be. Since we are considering perturbation theory, in which the weak coupling limits of $v_{\pm}, \lambda_{\pm} \rightarrow 0$ are assumed to exist, the powers of k and l should be restricted to

$$k \geq 0, \quad l \geq 0, \quad k + l \geq 1, \quad (5.12)$$

in order for $\Delta\Gamma_{\text{type-II}}^{\text{eff}}$ to be non-singular. Note that the functions f_{\pm} in (5.9) are non-singular in the weak coupling limits of $v_{\pm}, \lambda_{\pm} \rightarrow 0$ with (5.12).

Let us next clarify what kind of diagrams lead to the terms given in (5.10). To this end, we use the topological relation for Feynman diagrams

$$L = I - V_1 - V_3 + 1, \quad (5.13)$$

where L, I, V_1 and V_3 denote the numbers of loops, internal lines, v_{\pm} -vertices and λ_{\pm} -vertices of a Feynman diagram, respectively. Further, we have the relation

$$V_1 + 3V_3 = E + 2I, \quad (5.14)$$

because we have one-point vertices v_{\pm} and three-point vertices λ_{\pm} in the tree action. Here, E denotes the number of external lines. From (5.13) and (5.14), we obtain

$$L = -\frac{V_1}{2} + \frac{V_3}{2} - \frac{E}{2} + 1. \quad (5.15)$$

Since the values of V_1, V_3 and E for each term in (5.10) are given by $V_1 = k, V_3 = k + l - 1$ and $E = l + 1$, corresponding diagrams turn out to be tree ones without loops, i.e.

$$L = -\frac{k}{2} + \frac{k + l - 1}{2} - \frac{l + 1}{2} + 1 = 0. \quad (5.16)$$

Therefore, all the terms given in (5.10) have to be excluded from $\Delta\Gamma_{\text{type-II}}^{\text{eff}}$ because the effective potential $\Delta\Gamma_{\text{type-II}}^{\text{eff}}$ consists of 1PI diagrams only. This implies that there is no quantum correction to the tree type-II action $S_{\text{type-II}}$, i.e.

$$\Gamma_{\text{type-II}}^{\text{eff}} = S_{\text{type-II}}. \quad (5.17)$$

This completes the perturbative proof of the nonrenormalization theorem on the lattice.

In the essential point of our proof, any nilpotent-SUSY invariant local functional is classified into type-I or type-II. In the addition, possible type-II local functionals are only terms so called as CLR-type.⁶

5.2 Consideration of nonperturbative nonrenormalization property for type-II functional

We briefly consider a non-perturbative justification beyond a perturbative proof. Let us assume that no massless mode appears even non-perturbatively in the massive theory. The assumption leads us that our theory is sigularity-free at the origin in the coupling constant space with weak fields. In our quantum mechanical model, the assumption seems natural. But for higher dimensional cases, it may need more careful treatment.

Any way, from the assumption, the complex analysis in the previous subsection tells us that the holomorphic functions $f_{\pm}(z_{\pm}, w_{\pm})$ in (5.9) equals to zero in a neighborhood of the origin, since the perturbative proof in the previous subsection holds there. Then $f_{\pm}(z_{\pm}, w_{\pm})$ vanish identically on the *whole* complex plane because of the analytical continuation or the identity theorem in complex analysis. This suggests our nonrenormalization theorem holds nonperturbatively.

6. Summary and discussion

In this article, we have constructed a supersymmetric complex quantum mechanics model on lattice. The action is invariant under two nilpotent-SUSY transformations (Q_{\pm}) which form a maximal nilpotent subalgebra of full $N = 4$ SUSY and they keep a holomorphic property. Furthermore, they enable us to study Q_{\pm} cohomology exactly. As the result of the analysis, we can classify all local Q_{\pm} -invariant functionals with $N_F = 0$ into type-I (such as kinetic terms) and type-II (such as a mass term including the Wilson term and interaction terms.) The local functionals in a nilpotent-SUSY invariant effective action is also classified into the two types and we have proved nonrenormalization theorem for the type-II local functionals at *any order of perturbative expansion and without taking continuum limit, namely with finite lattice constant*. This means that we are able to realize more than one nilpotent-SUSY and the holomorphy even in a regularized theory and this is extremely nontrivial result. For nonperturbative justification of the nonrenormalization property of type-II functionals, we gave reasonable arguments.

The reasons why we succeeded to prove the theorem are (1) familiarity between two nilpotent-SUSY transformations and holomorphy, (2) definition of local functionals, (3) existence of the CLR, (4) cohomological analysis of the nilpotent SUSY. It should be noted that our model is quite similar to the $N = 1$, $D = 4$ Wess-Zumino model which has a F -term nonrenormalization theorem, kinetic terms suffering with quantum corrections.

⁶For massive perturbative calculations, major relations (5.13), (5.14), (5.15) and (5.16) are unchanged. We simply replace $f_{\pm}(w, z)$ with $f_{\pm}(w, z, m_+ m_-)$ in (5.9). As the result, we can obtain the same nonrenormalization theorem.

There remain several issues worth investigation such as higher-dimensional extension, co-homological analysis of local functionals with any fermion number and so on.

Acknowledgments

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A. Notations, H-representation for locality and some formulae

A lattice space coordinate is expressed by an integer as n , where $-N_L < n \leq N_L$ and $2N_L$ is the lattice size. The lattice constant a is set to unity. A translationally-invariant and local coefficient(TILC) is conveniently expressed by a holomorphic function, H-representation, for example

$$\tilde{A}(z_1, z_2, z_3) \equiv \sum_{k\ell m} A_{k\ell mn} z_1^{k-n} z_2^{\ell-n} z_3^{m-n}. \quad (\text{A.1})$$

The locality implies that $\tilde{A}(z_1, z_2, z_3)$ is holomorphic in a domain $\mathcal{D} = \{1 - \epsilon < |z_i| < 1 + \epsilon \mid \epsilon > 0, i = 1, 2, 3\}$. Although we refer [5] and [7] for the detailed arguments on locality, we note that the meaning of locality include not only ultralocality but also exponential damping. In this article, a terminology, *local functional* or *functional with locality*, used as a collection of fields with a local coefficient. A local difference operator ∇_{mn} in this paper is also a TILC

$$\nabla_{mn} = \nabla(m - n), \quad \tilde{\nabla}(z) \equiv \sum_m z^m \nabla(m), \quad (\text{A.2})$$

and $\tilde{\nabla}(z)$ is a holomorphic function in a domain $1 - \epsilon < |z| < 1 + \epsilon$ for small $\epsilon > 0$ with the property

$$\tilde{\nabla}(z = 1) = 0 \quad (\text{A.3})$$

which corresponds to its vanishing property for constant functions.

The symmbols $()$ and $\{\}$ for indices stand for symmetrization and anti-symmetrization, respectively. A hat symbol $\hat{}$ above an index in a sequence of indices means omission of the index. For examples, $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$, $A_{\{ab\}} = \frac{1}{2}(A_{ab} - A_{ba})$, $\hat{a}bc = bc$ and $ab\hat{c}d = abd$.

The hermitian conjugations of fields are defined as

$$\begin{cases} \phi_{\pm}^{\dagger} = \phi_{\mp}, \\ F_{\pm}^{\dagger} = F_{\mp}, \\ \chi_{\pm}^{\dagger} = \chi_{\mp}, \\ \bar{\chi}_{\pm}^{\dagger} = \bar{\chi}_{\mp}, \end{cases} \quad \begin{cases} \Phi_{\pm}^{\dagger} = \Phi_{\mp}, \\ \Upsilon_{\pm}^{\dagger} = \Upsilon_{\mp}, \\ \Psi_{\pm}^{\dagger} = \Psi_{\mp}, \\ S_{\pm}^{\dagger} = S_{\mp}, \\ \theta_{\pm}^{\dagger} = \theta_{\mp}. \end{cases} \quad (\text{A.4})$$

From this hermiticity, we can show the reality of our action with (2.22), (2.23), (2.26) ,

$$S \equiv S_0 + S_m + S_{int} = S^{\dagger}. \quad (\text{A.5})$$

We note that any two-point functions with translational invariance and locality commute with each other in the sense of matrices.

$$\begin{aligned} (AB)_{ik} &= \sum_j A_{ij} B_{jk} = \sum_j A(i-j) B(j-k) \\ &= \sum_{j'=i+k-j} B(i-j') A(j'-k) = \sum_{j'} B_{ij'} A_{j'k} = (BA)_{ik}. \end{aligned} \quad (\text{A.6})$$

Indeed, in the real lattice space, any translationally-invariant and local two-point functions including the difference operator ∇ and the massive propagator D commute with each other in the sense of matrices.

B. On a solution of a linear $\tilde{\nabla}$ equation for coefficients of functionals

In proving a fundamental theorem on cohomology of a nilpotent SUSY, we need a general solution of a linear equation for TILCs (translationally-invariant and local coefficients) with ∇ . In the linear equation of TILCs, a difference operator ∇ and the coefficient can be expressed by holomorphic functions with many-variables, namely holomorphic representation. The equation in question is typically

$$\sum_{a=1}^M \tilde{A}_a(z_1, \dots, z_N) \tilde{\nabla}(z_a) = 0, \quad (\text{B.1})$$

for $M \leq N$. Note that ∇ has two site-indices and the coefficient has $N+1$ site-indices in lattice site representation.

The solution of (B.1) can be generally written as

$$\tilde{A}_a(z_1, \dots, z_N) = \sum_{b=1}^M \tilde{A}_{\{ab\}}(z_1, \dots, z_N) \tilde{\nabla}(z_b), \quad \tilde{A}_{\{ab\}}(z_1, \dots, z_N) = -\tilde{A}_{\{ba\}}(z_1 \cdots z_N) \quad (\text{B.2})$$

where $\tilde{A}_{\{ab\}}$ for $a, b = 1, 2, \dots, M$ are holomorphic functions in the domain \mathcal{D}^N .

Proof. Let us write $z^{(0)}$ for the zero of $\tilde{\nabla}(z)$.⁷ We prove the above statement by induction. For $M=2$ case, (B.1) becomes

$$\tilde{A}_1(z_1, \dots, z_N) \tilde{\nabla}(z_1) + \tilde{A}_2(z_1, \dots, z_N) \tilde{\nabla}(z_2) = 0 \quad (\text{B.3})$$

where $N \geq 2$. By reminding $\tilde{\nabla}(z^{(0)}) = 0$ and setting $z_2 = z^{(0)}$ in (B.3), we obtain

$$\tilde{A}_1(z_1, z_2 = z^{(0)}, z_3, \dots, z_N) \tilde{\nabla}(z_1) = 0. \quad (\text{B.4})$$

Since (B.4) is true on a complex N -dimensional domain $\{(z_1, \dots, z_N) | z_1 \neq z^{(0)}\}$, the holomorphy (locality in other words) of the coefficient A_1 leads us to

$$\tilde{A}_1(z_1, z_2 = z^{(0)}, z_3, \dots, z_N) = 0 \quad (\text{B.5})$$

⁷Here we assume that the $\tilde{\nabla}(z)$ has a single zero $z^{(0)}$. For the case in which the $\tilde{\nabla}(z)$ has multiple zeros, e.g. in a doubling phenomena, by using interpolation methods such as Lagrange's one we can see the final result is unchanged.

and so

$$\tilde{A}_1(z_1, \dots, z_N) = \tilde{B}(z_1, \dots, z_N) \tilde{\nabla}(z_2) \quad (\text{B.6})$$

where \tilde{B} is a certain holomorphic function in the domain \mathcal{D}^N . From (B.3), (B.6) and the holomorphy of \tilde{A}_2 , we get

$$\tilde{A}_2(z_1, \dots, z_N) = -\tilde{B}(z_1, \dots, z_N) \tilde{\nabla}(z_1). \quad (\text{B.7})$$

From (B.6) and (B.7), the following equation

$$\tilde{A}_a(z_1, \dots, z_N) = \sum_{b=1}^2 \tilde{A}_{\{ab\}}(z_1, \dots, z_N) \tilde{\nabla}(z_b) \quad (\text{B.8})$$

is obtained with $\tilde{A}_{\{12\}} = \tilde{B} = -\tilde{A}_{\{21\}}$, $\tilde{A}_{\{11\}} = \tilde{A}_{\{22\}} = 0$ and we find that (B.2) indeed holds for $M = 2$.

Now let us assume that the above statement is true for $M = m$. Namely, for a linear equation

$$\sum_{a=1}^m \tilde{A}_a^{(m)}(z_1, \dots, z_N) \tilde{\nabla}(z_a) = 0 \quad (m < N), \quad (\text{B.9})$$

the general solution is written as

$$\tilde{A}_a^{(m)}(z_1, \dots, z_N) = \sum_{b=1}^m \tilde{A}_{\{ab\}}^{(m)}(z_1, \dots, z_N) \tilde{\nabla}(z_b), \quad \tilde{A}_{\{ab\}}^{(m)}(z_1, \dots, z_N) = -\tilde{A}_{\{ba\}}^{(m)}(z_1 \cdots z_N). \quad (\text{B.10})$$

Then we consider $M = m + 1$ case with $m + 1 < N$,

$$\sum_{a=1}^m \tilde{A}_a^{(m+1)}(z_1, \dots, z_N) \tilde{\nabla}(z_a) + \tilde{A}_{m+1}^{(m+1)}(z_1, \dots, z_N) \tilde{\nabla}(z_{m+1}) = 0. \quad (\text{B.11})$$

From a holomorphy of $\tilde{A}_{m+1}^{(m+1)}$ and

$$\tilde{A}_{m+1}^{(m+1)}(z_1 = z^{(0)}, \dots, z_m = z^{(0)}, z_{m+1}, \dots, z_N) = 0, \quad (\text{B.12})$$

$\tilde{A}_{m+1}^{(m+1)}(z_1, \dots, z_N)$ is shown to be expressed as

$$\tilde{A}_{m+1}^{(m+1)}(z_1, \dots, z_N) = \sum_{a=1}^m \tilde{B}_a^{(m+1)}(z_1, \dots, z_N) \tilde{\nabla}(z_a), \quad (\text{B.13})$$

where $\tilde{B}_a^{(m+1)}(a = 1, \dots, m)$ are some holomorphic functions.

As an example, we explain the above result in the case of $m = 2, N = 3$ below, although (B.13) generally holds. For $m = 2, N = 3$, (B.12) becomes

$$\tilde{A}_3^{(3)}(z_1 = z^{(0)}, z_2 = z^{(0)}, z_3) = 0, \quad (\text{B.14})$$

and we substitute this to the following identity

$$\begin{aligned}\tilde{A}_3^{(3)}(z_1, z_2, z_3) &= \frac{(\tilde{A}_3^{(3)}(z_1, z_2, z_3) - \tilde{A}_3^{(3)}(z_1 = z^{(0)}, z_2, z_3))}{\tilde{\nabla}(z_1)} \tilde{\nabla}(z_1) \\ &+ \frac{(\tilde{A}_3^{(3)}(z_1 = z^{(0)}, z_2, z_3) - \tilde{A}_3^{(3)}(z_1 = z^{(0)}, z_2 = z^{(0)}, z_3))}{\tilde{\nabla}(z_2)} \tilde{\nabla}(z_2) \\ &+ \tilde{A}_3^{(3)}(z_1 = z^{(0)}, z_2 = z^{(0)}, z_3)).\end{aligned}\quad (\text{B.15})$$

The result is

$$\begin{aligned}\tilde{A}_3^{(3)}(z_1, z_2, z_3) &= \frac{(\tilde{A}_3^{(3)}(z_1, z_2, z_3) - \tilde{A}_3^{(3)}(z_1 = z^{(0)}, z_2, z_3))}{\tilde{\nabla}(z_1)} \tilde{\nabla}(z_1) \\ &+ \frac{(\tilde{A}_3^{(3)}(z_1 = z^{(0)}, z_2, z_3) - \tilde{A}_3^{(3)}(z_1 = z^{(0)}, z_2 = z^{(0)}, z_3))}{\tilde{\nabla}(z_2)} \tilde{\nabla}(z_2),\end{aligned}\quad (\text{B.16})$$

Since $z_1 = z^{(0)}$ and $z_2 = z^{(0)}$ are removable singularities on the right hand side of (B.16), we obtain

$$\tilde{A}_3^{(3)}(z_1, z_2, z_3) = \sum_{a=1}^2 \tilde{B}_a^{(3)}(z_1, z_2, z_3) \tilde{\nabla}(z_a), \quad (\text{B.17})$$

where $\tilde{B}_a^{(3)}$ are holomorphic functions in \mathcal{D}^3 .

From (B.13), (B.11) is considered as an inhomogeneous linear equation for $\tilde{A}_a^{(m+1)}$. Thus, the general solution for the equation can be expressed as

$$\tilde{A}_a^{(m+1)}(z_1, \dots, z_N) = -\tilde{B}_a^{(m+1)}(z_1, \dots, z_N) \tilde{\nabla}(z_{m+1}) + \sum_{b=1}^m \tilde{A}_{\{ab\}}^{(m)}(z_1, \dots, z_N) \tilde{\nabla}(z_b), \quad (\text{B.18})$$

where we add the general solution for a homogeneous equation (B.9) to a special solution of (B.11). By defining

$$\begin{aligned}\tilde{A}_{\{ab\}}^{(m+1)}(z_1, \dots, z_N) &\equiv \tilde{A}_{\{ab\}}^{(m)}(z_1, \dots, z_N) \\ \tilde{A}_{\{m+1a\}}^{(m+1)}(z_1, \dots, z_N) &\equiv \tilde{B}_a^{(m+1)}(z_1, \dots, z_N) \\ \tilde{A}_{\{am+1\}}^{(m+1)}(z_1, \dots, z_N) &\equiv -\tilde{B}_a^{(m+1)}(z_1, \dots, z_N) \\ \tilde{A}_{\{m+1m+1\}}^{(m+1)}(z_1, \dots, z_N) &\equiv 0,\end{aligned}\quad (\text{B.19})$$

we obtain the general solution (B.2) for $M = m + 1$. □

By repeating the above argument, the following corollary is immediately obtained. If

$$\sum_{a_1=1}^M \tilde{A}_{\{a_1 \dots a_n\}}(z_1, \dots, z_N) \tilde{\nabla}(z_{a_1}) = 0, \quad (\text{B.20})$$

for $M \leq N$, then

$$\tilde{A}_{\{a_1 \dots a_n\}}(z_1, \dots, z_N) = \sum_{a_{n+1}=1}^M \tilde{A}_{\{a_1 \dots a_{n+1}\}}(z_1, \dots, z_N) \tilde{\nabla}(z_{a_{n+1}}). \quad (\text{B.21})$$

Here all \tilde{A} are holomorphic functions in a domain \mathcal{D}^N .

In applying these results to the proof of the fundamental theorem of δ_- -cohomology, we need the lattice site representation in which a linear equation for TILC

$$\sum_{a=1}^M \sum_m B_{k,m,n_1 \dots \hat{n}_a \dots n_N} \nabla_{m,n_a} = 0 \quad (\text{B.22})$$

has a general translationally invariant and local solution

$$B_{k,m,n_1 \dots \hat{n}_a \dots n_N} = \sum_{b=1, b \neq a}^M \sum_{m'} C_{k,\{mm'\},n_1 \dots \hat{n}_a \dots \hat{n}_b \dots n_N} \nabla_{m',n_b}, \quad (\text{B.23})$$

where C 's are some TILC. Although the index k seems redundant, at least one extra index is necessary for writing down corresponding lattice site representation, since the number of independent indices are one less than the total number of indices for the translationally invariant coefficient.

C. Proof of fundamental theorem on the cohomology of nilpotent SUSY

Since the analysis of $-$ -type fields is similar to that of $+$ -type fields, we concentrate on only $+$ fields ($+$ -type functionals) and Q_- in this appendix. Thus, we omit the subscript of fields as $\phi(= \phi_+)$, $\chi(= \chi_+)$, $F(= F_+)$, $\bar{\chi}(= \bar{\chi}_+)$. We set the $U(1)_R$ charge R of \mathcal{O} as $R = n_+ - 1 - 2K$ where n_+ and K are the number of $+$ -type fields and the combined number of F and $\bar{\chi}$, respectively. When $K = 0$, namely $R = R_{\max} \equiv n_+ - 1$, the functional $\mathcal{O} = \sum C \chi \phi \dots \phi$ is called as a CLR term with $N_F = 1$. The N is defined as the total number of ϕ and χ . K and N are conserved under Q_- transformation (2.4).

Fundamental theorem of Q_- -cohomology says if $Q_- \mathcal{O} = 0$ for a $+$ -type local functional \mathcal{O} in a sector of $N_F = 1$, in the case of $R < R_{\max}$, $\mathcal{O} = Q_- \mathcal{P}$ with a local functional \mathcal{P} , and in the case of $R = R_{\max}$, \mathcal{O} can be a CLR term with $N_F = 1$ up to some Q_- -exact local functionals.

This theorem leads us that CLR terms ($R = R_{\max}$) are only nontrivial cohomology candidates in the fermion number 1 sector. Namely, it is a fundamental theorem on cohomology of nilpotent SUSY. This theorem can be proved for both original fields and superfields although we carry out for original fields here.

Proof. Since K, N are conserved numbers under Q_- -transformation, it is sufficient to consider the following functionals as fermion number $+1$ translationally invariant, local and general functionals

$$\begin{aligned} \mathcal{O}(K, N) = & \sum_{\mathbf{k}, \ell, \mathbf{m}, \mathbf{n}} \sum_{p=0}^{\min(K, N)} B_{(k_1 \dots k_{K-p})\{\ell_1 \dots \ell_p\}\{m_1 \dots m_{p+1}\}(n_1 \dots n_{N-p})}^{(p)} \\ & \times F_{k_1} \dots F_{k_{K-p}} \bar{\chi}_{\ell_1} \dots \bar{\chi}_{\ell_p} \chi_{m_1} \dots \chi_{m_{p+1}} \phi_{n_1} \dots \phi_{n_{N-p}}. \end{aligned} \quad (\text{C.1})$$

where $B^{(p)}$ is a TILC and bold indices $\mathbf{k}, \mathbf{\ell}, \mathbf{m}, \mathbf{n}$ stand for multi indices

$$\begin{aligned}\mathbf{k} &\equiv k_1, \dots, k_{K-p}, & \mathbf{\ell} &\equiv \ell_1, \dots, \ell_p, \\ \mathbf{m} &\equiv m_1, \dots, m_{p+1}, & \mathbf{n} &\equiv n_1, \dots, n_{N-p}.\end{aligned}\tag{C.2}$$

$B^{(\min K, N)+1} \equiv 0$ is set for convenience, We impose the Q_- -invariant (closed form) condition

$$Q_- \mathcal{O}(K, N) = 0 \tag{C.3}$$

on \mathcal{O} . Using a single dot-symbol for absence of the corresponding index such as $\{\cdot\}$, the condition (C.3) is translated into the condition on the TILCs,

$$\sum_{j=1}^{N+1} \sum_m B_{(k_1 \dots k_K) \{\cdot\} \{m\} (n_1 \dots \hat{n}_j \dots n_{N+1})}^{(0)} \nabla_{mn_j} = 0, \tag{C.4}$$

for $p = 0$ and

$$\begin{aligned}& \frac{p+1}{N-p+1} \sum_{j=1}^{N-p+1} \sum_m B_{(k_1 \dots k_{K-p}) \{\ell_1 \dots \ell_p\} \{m_1 \dots m_p m\} (n_1 \dots \hat{n}_j \dots n_{N-p+1})}^{(p)} \nabla_{mn_j} \\ & + \frac{K-p+1}{p} \sum_{j=1}^p (-1)^j \sum_k B_{(k_1 \dots k_{K-p} k) \{\ell_1 \dots \hat{\ell}_j \dots \ell_p\} \{m_1 \dots m_p\} (n_1 \dots n_{N-p+1})}^{(p-1)} \nabla_{k\ell_j} = 0, \end{aligned} \tag{C.5}$$

for $1 \leq p \leq \min(K, N)$, where we used a hat ($\hat{\cdot}$) symbol for absent index. For $p = \min(K, N) + 1 = N + 1$ ($K > N$ case), (C.5) becomes

$$\sum_{j=1}^{N+1} \sum_k (-1)^j B_{(k_1 \dots k_{K-N-1} k) \{\ell_1 \dots \hat{\ell}_j \dots \ell_{N+1}\} \{m_1 \dots m_{N+1}\} (\cdot)}^{(N)} \nabla_{k\ell_j} = 0. \tag{C.6}$$

In the case of $p = \min(K, N) + 1 = K + 1$ ($K \leq N$ case), there is no extra condition such as (C.6) for $B^{(K)}$, since $B^{(K+1)} = 0$ and $K - p + 1 = 0$ in (C.5). Note that (C.5) can be solved as inhomogeneous linear equations for $B^{(p)}$ except for $K = 0$.

Then we discuss solutions of the above conditions for $K = 0$ and $K \geq 1$ cases separately:

1. $K \geq 1$ case

To express a general solution, we introduce the following TILCs

$$C_{(k_1 \dots k_{K-p+1}) \{\ell_1 \dots \ell_{p-1}\} \{m_1 \dots m_{p+1}\} (n_1 \dots n_{N-p})}^{\{p\}}, \tag{C.7}$$

and

$$C^{\{0\}} = 0. \tag{C.8}$$

From the symmetric property of $C^{\{p\}}$, the coefficient satisfies the following properties,

$$\sum_{m, n} \sum_{i, j=1, i \neq j}^{N-p+2} C_{(k_1 \dots k_{K-p+1}) \{\ell_1 \dots \ell_{p-1}\} \{m_1 \dots m_{p-1} mn\} (n_1 \dots \hat{n}_i \dots \hat{n}_j \dots n_{N-p+2})}^{\{p\}} \nabla_{mn_i} \nabla_{nn_j} = 0, \tag{C.9}$$

$$\sum_{k,\ell} \sum_{i,j=1,i \neq j}^{p+1} (-1)^{i+j} C_{(k_1 \dots k_{K-p-1} k \ell) \{\ell_1 \dots \hat{\ell}_i \dots \ell_{p+1}\} \{m_1 \dots m_{p+1}\} (n_1 \dots n_{N-p})}^{\{p\}} \nabla_{k \ell_i} \nabla_{\ell \ell_j} = 0, \quad (\text{C.10})$$

$$\begin{aligned} & \sum_{k,m} \sum_{i=1}^{N-p} \sum_{j=1}^{p+1} (-1)^j C_{(k_1 \dots k_{K-p-1} k) \{\ell_1 \dots \hat{\ell}_j \dots \ell_{p+1}\} \{m_1 \dots m_{p+1} m\} (n_1 \dots \hat{n}_i \dots n_{N-p})}^{\{p+1\}} \nabla_{k \ell_j} \nabla_{m n_i} \\ &= \sum_{k,m} \sum_{i=1}^{p+1} \sum_{j=1}^{N-p} (-1)^i C_{(k_1 \dots k_{K-p-1} k) \{\ell_1 \dots \hat{\ell}_i \dots \ell_{p+1}\} \{m_1 \dots m_{p+1} m\} (n_1 \dots \hat{n}_j \dots n_{N-p})}^{\{p+1\}} \nabla_{m n_j} \nabla_{k \ell_i}. \end{aligned} \quad (\text{C.11})$$

From a solution (B.23), the general solution of TILC $B^{(0)}$ for (C.4) is written as

$$B_{(k_1 \dots k_K) \{\cdot\} \{m\} (n_1 \dots \hat{n}_j \dots n_{N+1})}^{(0)} = \sum_{i=1,i \neq j}^{N+1} \sum_{m'} C_{(k_1 \dots k_K) \{\cdot\} \{m m'\} (n_1 \dots \hat{n}_i \dots \hat{n}_j \dots n_{N+1})}^{\{1\}} \nabla_{m' n_i}. \quad (\text{C.12})$$

A general solution for $B^{(1)}$ and $B^{(2)}$ are

$$\begin{aligned} & B_{(k_1 \dots k_{K-1}) \{\ell\} \{m_1 m_2\} (n_1 \dots n_{N-1})}^{(1)} \\ &= \frac{KN}{1 \cdot 2} \sum_{k'} C_{(k_1 \dots k_{K-1} k') \{\cdot\} \{m_1 m_2\} (n_1 \dots n_{N-1})}^{\{1\}} \nabla_{k' \ell} \\ &+ \sum_{m'} \sum_{j=1}^{N-1} C_{(k_1 \dots k_{K-1}) \{\ell\} \{m_1 m_2 m'\} (n_1 \dots \hat{n}_j \dots n_{N-1})}^{\{2\}} \nabla_{m' n_j} \end{aligned} \quad (\text{C.13})$$

and

$$\begin{aligned} & B_{(k_1 \dots k_{K-2}) \{\ell_1 \ell_2\} \{m_1 m_2 m_3\} (n_1 \dots n_{N-2})}^{(2)} \\ &= -\frac{(K-1)(N-1)}{2 \cdot 3} \sum_{k'} \sum_{i=1}^2 (-1)^i C_{(k_1 \dots k_{K-2} k') \{\ell(\hat{\ell}_i)\} \{m_1 m_2 m_3\} (n_1 \dots n_{N-2})}^{\{2\}} \nabla_{k' \ell_i} \\ &+ \sum_{m'} \sum_{j=1}^{N-2} C_{(k_1 \dots k_{K-2}) \{\ell_1 \ell_2\} \{m_1 m_2 m_3 m'\} (n_1 \dots \hat{n}_j \dots n_{N-2})}^{\{3\}} \nabla_{m' n_j}, \end{aligned} \quad (\text{C.14})$$

where we used (B.23), (C.9), (C.10), and (C.11) and $\ell(\hat{\ell}_i)$ means ℓ_2 for $i = 1$ and ℓ_1 for $i = 2$. Consequently, we get a general solution for $B^{(p)}$

$$\begin{aligned} & B_{(k_1 \dots k_{K-p}) \{\ell_1 \dots \ell_p\} \{m_1 \dots m_{p+1}\} (n_1 \dots n_{N-p})}^{(p)} \\ &= -\frac{(K-p+1)(N-p+1)}{p(p+1)} \sum_{k'} \sum_{i=1}^p (-1)^i C_{(k_1 \dots k_{K-p} k') \{\ell_1 \dots \hat{\ell}_i \dots \ell_p\} \{m_1 \dots m_{p+1}\} (n_1 \dots n_{N-p})}^{\{p\}} \nabla_{k' \ell_i} \\ &+ \sum_{m'} \sum_{j=1}^{N-p} C_{(k_1 \dots k_{K-p}) \{\ell_1 \dots \ell_p\} \{m_1 \dots m_{p+1} m'\} (n_1 \dots \hat{n}_j \dots n_{N-p})}^{\{p+1\}} \nabla_{m' n_j}. \end{aligned} \quad (\text{C.15})$$

There remains an extra condition (C.6). For $K > N$ and $p = N$, the condition (C.6) reads

$$\begin{aligned} & B_{(k_1 \dots k_{K-N})\{\ell_1 \dots \ell_N\}\{m_1 \dots m_{N+1}\}}^{(N)}(\cdot) \\ &= -\frac{(K-N+1)}{N(N+1)} \sum_{k'} \sum_{i=1}^N (-1)^i C_{(k_1 \dots k_{K-N} k')\{\ell_1 \dots \hat{\ell}_i \dots \ell_N\}\{m_1 \dots m_{N+1}\}}^{\{N\}}(\cdot) \nabla_{k' \ell_i}. \end{aligned} \quad (\text{C.16})$$

By directly solving (C.4), we also have

$$B^{(0)} = 0 \quad (\text{C.17})$$

for $N = 0$. It is consistent with $N = 0$ case of (C.16).

Finally, from (C.1), (C.15) and (C.16), we can generally obtain an exact form

$$\begin{aligned} \mathcal{O}(K, N) = Q_- \Big(\sum_{\mathbf{k}, \ell, \mathbf{m}', \mathbf{n}'}^{\min(K, N-1)} \sum_{p=0}^{\min(K, N-1)} \frac{i(N-p)}{(p+2)} C_{(\mathbf{k})\{\ell\}\{\mathbf{m}'\}(\mathbf{n}')}^{\{p+1\}} \\ \times F_{k_1} \dots F_{k_{K-p}} \bar{\chi}_{\ell_1} \dots \bar{\chi}_{\ell_p} \chi_{m_1} \dots \chi_{m_{p+2}} \phi_{n_1} \dots \phi_{n_{N-p-1}} \Big). \end{aligned} \quad (\text{C.18})$$

where $\mathbf{m}' \equiv m_1, \dots, m_{p+2}$, $\mathbf{n}' \equiv n_1, \dots, n_{N-p-1}$. In $N = 0$ case, there is no Q_- -invariant translationally invariant and local functional, which is consistent with (C.17).

2. $K = 0$ case

In $K = 0$, the functional can be written as

$$\mathcal{O}(0, N) = \sum_{m_1, n_1, \dots, n_N} B_{(\cdot)\{\cdot\}\{m_1\}(n_1 \dots n_N)}^{(0)} \chi_{m_1} \phi_{n_1} \dots \phi_{n_N} \quad (\text{C.19})$$

and the Q_- -invariant condition corresponds to a cyclic Leibniz rule (CLR)

$$\sum_{a=1}^{N+1} \sum_{n_1, \dots, n_{N+1}} B_{(\cdot)\{\cdot\}\{n_a\}(n_1 \dots n_{a-1} n_{a+1} \dots n_{N+1})}^{(0)} \phi_{n_1} \dots \phi_{n_{a-1}} (\nabla \phi)_{n_a} \phi_{n_{a+1}} \dots \phi_{n_{N+1}} = 0. \quad (\text{C.20})$$

This term (C.19) is just what is called *CLR term with $N_F = 1$* in section 3. For the coefficient $B^{(0)}$ to be proportional to ∇ , at least one irrelevant index is necessary, just as the last comment of appendix B. But all $N + 1$ indices of $B^{(0)}$ are relevant for the contraction with index of ∇ , thereby the previous argument does not apply here. So it implies that $B^{(0)}$ in (C.19) is not always written like (C.12), i.e. exact form in $K = 0$.

□

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